Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2014–03–12, 14:00-18:00 V Halls

Telephone: Mohammad Asadzadeh: 0703-088304
Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p- For GU students**G**:15-24p, **VG**: 25p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1314/index.html

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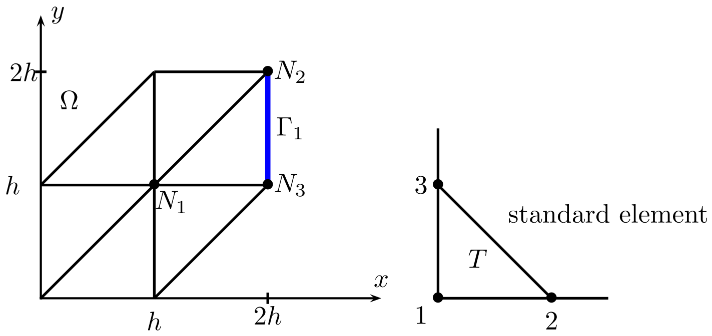
1. Let v be a continuously differentiable function on the interval $(0, b)$ and $\|\cdot\|$ denotes the $L_2(0, b)$-norm. Show the following version of the Poincare inequality:

$$\|v\|^2 \le b\Big(v(0)^2 + v(b)^2 + b\|v'\|^2\Big). \tag{1}$$

Hint: use integration by parts for $\int_0^{b/2} v^2(x)\,dx$ and $\int_{b/2}^{b} v^2(x)\,dx$, and note that $\frac{d}{dx}(x - b/2) = 1$.

2. Let Ω be the hexagonal domain with the uniform triangulation as in the figure below. Compute



the stiffness matrix and the load vector for the cG(1) approximate solution for the problem:

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega, \\ \partial u/\partial x = 0, & (x, y) \in \Gamma_1 := \{(x, y) \in \partial\Omega : x = 2h,\ h \le y \le 2h\}, \\ u = 0, & \text{on } \partial\Omega \setminus \Gamma_1. \end{cases} \tag{2}$$

3. Let $0 < \alpha(x) \le K$ for $x \in [0, 1]$, where K is a constant. Derive an *a priori* and an *a posteriori* error estimate for the cG(1) finite element method for the problem

$$-u''(x) + \alpha(x)u(x) = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0, \tag{3}$$

in the energy norm: $||e||_E^2 = ||e'||^2 + ||\sqrt{\alpha}\, e||^2$. How does a priori error bound depend on K?

4. Let ε be a positive constant, $\alpha(x) \ge 0$ and $\alpha'(x) \le 0$. Consider the boundary value problem

$$-\varepsilon u'' + \alpha(x)u' + u = f(x), \quad 0 < x < 1, \qquad u(0) = 0, \quad u'(1) = 0, \tag{4}$$

Show, the following L_2-stability estimates:

$$\sqrt{\varepsilon}||u'|| \le C_1||f||, \quad ||\alpha u'|| \le C_2||f||, \quad \varepsilon||u''|| \le C_3||f||, \qquad \text{with } ||w|| = \Big(\int_0^1 w^2\,dx\Big)^{1/2}.$$

5. Formulate and prove the Lax-Milgram theorem for symmetric scalar products (i.e. give the conditions on linear and bilinear forms and derive the proof of the Riesz representation theorem).
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1. The assertion follows from the following elementary chain of calculus:

$$\begin{aligned}
||v||^2_{L_2(0,b)} &= \int_0^b v^2(x)\,dx = \int_0^{b/2} v^2(x)\,dx + \int_{b/2}^b v^2(x)\,dx \\
&= [(x-b/2)v^2(x)]_0^{b/2} + [(x-b/2)v^2(x)]_{b/2}^b - \int_0^b (x-b/2)2v(x)v'(x)\,dx \\
&\leq \frac{b}{2}v(0)^2 + \frac{b}{2}v(b)^2 + b||v||||v'|| \leq \frac{b}{2}v(0)^2 + \frac{b}{2}v(b)^2 + \frac{b^2}{2}||v'||^2 + \frac{1}{2}||v||^2.
\end{aligned}$$

2. Let V be the linear function space defined by

$$V := \{v : v \in H^1(\Omega),\ \ v = 0,\ \text{ on } \partial\Omega \setminus \Gamma_1\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (1, v), \qquad \forall v \in V.$$

Now using Green's formula and the fact that $v = 0$ on $\partial\Omega \setminus \Gamma_1$, we have that

$$\begin{aligned}
-(\Delta u, \nabla v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v\,ds \\
&= (\nabla u, \nabla v) - \int_{\partial\Omega\setminus\Gamma_1} (n \cdot \nabla u) v\,ds - \int_{\Gamma_1} (n \cdot \nabla u) v\,ds \\
&= (\nabla u, \nabla v) - \int_{\Gamma_1} (n \cdot \nabla u) v\,ds = (\nabla u, \nabla v), \qquad \forall v \in V,
\end{aligned}$$

where in the last step we have that $n\|_{\Gamma_1} = (1, 0)$, thus $n \cdot \nabla u = u_x = 0$ on Γ_1. Hence, the variational formulation is:

$$(\nabla u, \nabla v) = (1, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega \setminus \Gamma_1$: Then, the $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) = (1, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^3 \xi_j \varphi_j(x)$, where φ_j are the standard basis functions (φ_1 is the basis function for the interior node N_1 and φ_2 and φ_3 are corresponding basis functions for the boundary nodes N_1 and N_2, respective) we obtain the system of equations

$$\sum_{j=1}^3 \xi_j \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\,dx = \int_\Omega f\varphi_i\,dx, \quad i = 1, 2, 3.$$

In matrix form this can be written as $S\xi = F$, where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, and $F_i = (f, \varphi_i)$ is the load vector.

We first compute the stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1,x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1,x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\phi_2(x_1,x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\phi_3(x_1,x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$, we can easily compute

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\, dx = \frac{2}{h^2}|T| = 1,$$
$$s_{12} = s_{21} = (\nabla\phi_1, \nabla\phi_2) = \int_T \frac{-1}{h^2}|T| = -1/2,$$
$$s_{23} = s_{32} = (\nabla\phi_2, \nabla\phi_3) = 0,$$
$$s_{22} = s_{33} = \ldots = \frac{1}{h^2}|T| = 1/2.$$

Thus by symmetry we get that the local stiffness matrix for the standard element is:

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global stiffness matrix S from the local stiffness matrix s:

$$S_{11} = 2s_{11} + 4s_{22} = 2 + 2 = 4, \qquad S_{12} = S_{21} = s_{23} = 0 \qquad S_{13} = s_{12} = -1/2$$
$$S_{22} = s_{22} = 1/2 \qquad S_{23} = s_{12} = -1/2, \qquad S_{33} = s_{11} = 1/2.$$

The remaining matrix elements are obtained by symmetry $S_{ij} = S_{ji}$. Hence,

$$S = \frac{1}{2}\begin{bmatrix} 8 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

As for the load vector we have that

$$\int_\Omega \varphi_1 = 6\frac{1}{3}\frac{h^2}{2}.1 = h^2, \qquad \int_\Omega \varphi_2 = \int_\Omega \varphi_3 = \frac{1}{3}\frac{h^2}{2}.1 = \frac{h^2}{6}.$$

This the load vector is given by $b = h^2(1, 1/6, 1/6)^t$. Observe that, here S has become independent of h.

3. We multiply the differential equation by a test function $v \in H_0^1 = \{v : ||v|| + ||v'|| < \infty,\ v(0) = v(1) = 0\}$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + \alpha uv) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{5}$$

Then, the cG(1) *Finite Element Method* reads as follows: Find $U \in V_h^0$ such that

$$\int_I (U'v' + \alpha Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{6}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Let now $e = u - U$, then (5)- (6) gives that

$$\int_I (e'v' + \alpha ev) = \quad \forall v \in V_h^0, \text{(Galerkin Orthogonality)}. \tag{7}$$

A posteriori error estimate: We use again ellipticity (**??**), Galerkin orthogonality (7), and the variational formulation (5) to get

$$
\begin{aligned}
\|e\|_E^2 &= \int_I (e'e' + \alpha ee) = \int_I ((u-U)'e' + \alpha(u-U)e) = \{v = e \text{in}(5)\} \\
&= \int_I fe - \int_I (U'e' + \alpha Ue) = \{v = \pi_h e \text{in}(6)\} \\
&= \int_I f(e-\pi_h e) - \int_I (U'(e-\pi_h e)' + \alpha U(e - \pi_h e)) = \int_I R(U)((e-\pi_h e).
\end{aligned} \tag{8}
$$

where $R(U) = f + U'' - \alpha U = f - \alpha U$ (since $U'' \equiv 0$ for $U \in V_h^0$). Further in the last equality we use partial integration and the fact that $e(x_j) = (\pi e)(x_j)$, for j:s being the node points. Thus Hence, (8) yields:

$$
\begin{aligned}
\|e\|_E^2 &\le C\|hR(U)\|_{L_2(I)}\|h^{-1}(e-\pi_h e)\|_{L_2(I)} \le C_i\|hR(U)\|_{L_2(I)}\|e'\|_{L_2(I)} \\
&\le C_i\|hR(U)\|_{L_2(I)}\|e\|_E.
\end{aligned} \tag{9}
$$

Consequently we have the a posteriori error estimate

$$
\|e\|_E \le C_i\|hR(U)\|_{L_2(I)}. \tag{10}
$$

A priori error estimate: We use a short hand notation, viz:

$$
(v,w)_E = \int_I (v'w' + \alpha vw)\,dx, \qquad \text{and } \|v\|_E^2 = (v,v)_E = \int_I (v'^2 + \alpha v^2). \tag{11}
$$

Thus, by the Galerkin orthogonality reads as

$$
(e,v)_E = 0, \quad \forall v \in V_h^0. \tag{12}
$$

Hence, we compute using (12) with $v = U - \pi_h u$, with $\pi_h u$ being the interpolant of u, that

$$
\begin{aligned}
\|e\|_E^2 &= (e,e)_E = (e, u-U)_E = (e, u-\pi_h u)_E - (e, U - \pi_h u)_E = (e, u - \pi_h u)_E \\
&\le \|e\|_E\|u - \pi_h u\|_E,
\end{aligned} \tag{13}
$$

where in the last step we used the Cauchy-Schwarz inequality. This gives that

$$
\|e\|_E \le \|u - \pi_h u\|_E. \tag{14}
$$

But for the interpolation error we have that

$$
\begin{aligned}
\|u - \pi_h u\|_E^2 &= \|(u-\pi_h u)'\|_E^2 + \|\sqrt{\alpha}(u - \pi_h u)\|_E^2 \\
&\le C_i^2\|hu''\|_{L_2(I)}^2 + C_i^2 K\|h^2 u''\|_{L_2(I)}^2.
\end{aligned} \tag{15}
$$

This yields the a priori error estimate , viz

$$
\|e\|_E \le C_i\Big(\|hu''\|_{L_2(I)} + \sqrt{K}\|h^2 u''\|_{L_2(I)}\Big). \tag{16}
$$

4. Multiplication by u gives

$$
\varepsilon||u'||^2 + \int_0^1 \alpha u' u\,dx + ||u||^2 = (f,u) \le ||f||||u|| \le \frac{1}{2}||f||^2 + \frac{1}{2}||u||^2.
$$

Here

$$
\begin{aligned}
\int_0^1 \alpha u' u\,dx &= \frac{1}{2}\int_0^1 \alpha \frac{d}{dx}u^2\,dx \\
&= \frac{1}{2}\alpha(1)u(1)^2 - \frac{1}{2}\int_0^1 \alpha' u^2\,dx \ge 0,
\end{aligned} \tag{17}
$$

and hence

$$
\varepsilon||u'||^2 + \frac{1}{2}||u||^2 \le \frac{1}{2}||f||^2.
$$

This proves

$$
\sqrt{\varepsilon}||u'|| \le ||f||, \qquad ||u|| \le ||f||. \tag{18}
$$

Multiply the equation by $\alpha u'$ and integrate over x to obtain

$$-\varepsilon \int_0^1 u''\alpha u'\,dx + ||\alpha u'||^2 + \int_0^1 \alpha u' u\,dx \leq \frac{1}{2}||f||^2 + \frac{1}{2}||\alpha u'||^2.$$

Hence by (11)

$$\begin{aligned} ||\alpha u'||^2 &\leq ||f||^2 + \varepsilon \int_0^1 \alpha \frac{d}{dx}(u')^2\,dx \\ &= ||f||^2 - \varepsilon\alpha(0)u'(0)^2 - \varepsilon \int_0^1 \alpha'(u')^2\,dx \\ &\leq ||f||^2 + ||\alpha'||\varepsilon||u'||^2 \leq ||f||^2 + C\varepsilon||u'||^2. \end{aligned}$$

Using also (12) we conclude

$$||\alpha u'|| \leq C||f||. \tag{19}$$

Finally, by the differential equation and (12) and (14) we get

$$\varepsilon||u''|| = ||f - \alpha u' - u|| \leq ||f|| + ||\alpha u'|| + ||u|| \leq C||f||.$$

5. See the Book and/or Lecture Notes.

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