Mathematic Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2013–08–28, 8:30-12:30 V Halls

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p- For GU students**G**:15-24p, **VG**: 25p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1213/index.html

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1. Derive the cG(1)-cG(1), Crank-Nicolson approximation, for the initial boundary value problem

$$(1) \qquad \begin{cases} \dot{u} - u'' = f, & 0 < x < 1, \quad t > 0, \\ u'(0,t) = u'(1,t) = 0, & u(x,0) = 0, \quad x \in [0,1], \quad t > 0, \end{cases}$$

2. Consider the following boundary value problem:

$$(2) \qquad -(\alpha u')' + \beta u' + \gamma u = f, \quad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

with the corresponding variational formulation

$$(3) \qquad a(u, \varphi) = L(\varphi), \qquad \forall \varphi \in H_0^1.$$

Show that if $\alpha(x) \geq \alpha_0 > 0$, and $\gamma(x) - \beta'(x)/2 \geq 0$, for $x \in I = [0,1]$, then (2) admits a unique solution $u \in H_0^1$ satisfying the stability estimate

$$||u||_1 \leq \frac{2}{\alpha_0} ||f||.$$

3. Consider the boundary value problem

$$-(au')' = f, \quad 0 < x < 1, \qquad u(0) = u'(1) = 0.$$

(a) Show that the solution of this problem minimizes the energy integral

$$F(v) = \frac{1}{2} \int_0^1 a(v')^2 - \int_0^1 fv,$$

i.e., we have that $u \in V$ where V is some function space and $F(u) = \min_{v \in V} F(v)$.
(b) Show that for $a = 1$, and for a corresponding discrete minimum: $F(U) = \min_{v \in V_h} F(v)$, with $U \in V_h \subset V$, we have that

$$F(U) = F(u) + \frac{1}{2} ||(u - U)'||^2.$$

(c) Let $a = 1$ and show an a posteriori error estimate for the discrete energy minimum: i.e., for $|F(U) - F(u)|$, with V_h being the space of piecewise linear functions on subintervals of length h.

4. Consider the Poisson equation with the Neumann boundary condition:

$$(4) \qquad -\Delta u = f, \quad \text{in } \Omega \in \mathbf{R}^2, \qquad \text{with} \qquad -\mathbf{n} \cdot \nabla u = k\, u, \ \text{on } \partial\Omega,$$

where $k > 0$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ ($\partial\Omega$ is the boundary of Ω).
a) Prove the Poincare inequality: $\|u\|_{L_2(\Omega)} \leq C_\Omega(\|u\|_{L_2(\partial\Omega)} + \|\nabla u\|_{L_2(\Omega)})$.
b) Use the inequality in a) and show that $\|u\|_{L_2(\partial\Omega)} \to 0$ as $k \to \infty$.

5. Let U be the continuous piecewise linear finite element approximation of the two point boundary value problem

$$-(a(x)u'(x))' = f(x) \quad 0 < x < 1, \qquad u(0) = u(1) = 0, \qquad a(x) > 0.$$

Prove the following a posteriori error estimate (C_i is an interpolation constant):

$$\|u' - U'\|_a \leq C_i \|h\, R(U)\|_{a^{-1}}.$$

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1. Make the cG(1)-cG(1) ansatz

$$U(x,t) = U_{n-1}(x)\psi_{n-1}(t) + U_n(x)\psi_n(t), \qquad \text{with} \quad U_n(x) = \sum_{j=1}^{M} U_{n,j}\varphi_j(x),$$

in the variational formulation

$$\int_{I_n}\int_0^1 u'v' = \int_{I_n}\int_0^1 f\,v, \qquad I_n = (t_{n-1}, t_n).$$

Recall that $v = \varphi_j(x)$, $j = 1, \dots, M$ and

$$\psi_{n-1}(t) = \frac{t_n - t}{t_n - t_{n-1}}, \qquad \psi_n(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}.$$

For a uniform tile partition with $k := t_n - t_{n-1}$, this yields the equation system

$$(M + \frac{k}{2}S)U_n = (M - \frac{k}{2}S)U_{n-1} + k\mathbf{b}_n.$$

Here U_n is the node-vale vector with entries $U_{n,j}$, M is the mass-matrix with elements $\int_0^1 \varphi_i(x)\varphi_j(x)$, S is the stiffness-matrix with elements $\int_0^1 \varphi_i'(x)\varphi_j'(x)$, and $\mathbf{b}_n$ is the load vector with elements $\frac{1}{k}\int_{I_n}\int_0^1 f\varphi_i(x)$. The corresponding dG0 ($\approx$ implicit Euler) time-stepping yields

$$(M + kS)U_n = MU_{n-1} + k\mathbf{b}_n.$$

2. The variational formulation would be

$$(5) \qquad a(u,\varphi) := \int_0^1 (\alpha u'\varphi' + \beta u'\varphi + \gamma u\varphi)\,dx = \int_0^1 f\varphi\,dx = L(\varphi), \quad \forall \varphi \in C_0^1.$$

Note that by Cauchy-Schwarz inequality we get the Poincare's inequality:

$$\|v\| \le \|v'\|,$$

which gives

$$(6) \qquad \|v\|_1 = \Big(\|v\|^2 + \|v'\|^2\Big)^{1/2} \le \sqrt{2}\|v'\|^2, \quad \forall v \in H_0^1.$$

Furthermore by the assumptions

$$\int_0^1 (\beta v'v + \gamma v^2\,dx = \Big[\frac{1}{2}\beta v^2\Big]_0^1 + \int_0^1 (\gamma - \frac{1}{2}\beta')v^2\,dx \ge 0, \quad \forall v \in H_0^1.$$

Now using (6) and the assumptions we have that

$$a(v,v) \ge \min_{x\in\Omega} \alpha(x)\|v'\|^2 \ge \frac{\alpha_0}{2}\|v\|_1^2. \quad \forall v \in H_0^1.$$

Thus $a(\cdot,\cdot)$ is coercive in H_0^1. Moreover, estimating the coefficients in the (5) by their maxima and using the Cauchy-Schwartz inequality. we have

$$|a(v,w)| \le C\int_0^1 \Big(|v'w'| + |v'w| + |vw|\Big)\,dx \le C\|v\|_1\|w\|_1.$$

we have that the bilinear form $a(v,w)$ is bounded in H_0^1. Now since $L(\cdot)$ is also bounded in H_0^1:

$$|L(\varphi)| = |(f,\varphi)| \le \|f\|\|\varphi\|_1, \quad \forall \varphi \in H_0^1,$$

we have using Lax-Milgram lemma that the (5) admits a unique solution.

Finally

$$\frac{\alpha_0}{2}\|u\|_1^2 \le a(u,u) = (f,u) \le \|f\|\|u\| \le \|f\|\|u\|_1,$$

proves the last statement, that:

$$\|u\|_1 \le \frac{2}{\alpha_0}\|f\|.$$

3. (a) See lecture notes, chapter 8, page 8.3 (the only modification is that you put $g_1 = 0$). Thus from the differential equation for u it follows, after multiplication by w and using integration by parts, that

$$\int_0^1 au'w'\,dx = \int_0^1 fw\,dx. \tag{7}$$

Hence, for arbitrary $v = u + w$ we have that

$$F(v) = F(u+v) = F(u) + \int_0^1 au'w'\,dx - \int_0^1 fw\,dx + \int_0^1 a(w')^2\,dx \ge F(u), \tag{8}$$

since using (1) the first two integrals are add up to zero and the third integral is ≥ 0.
(b) Let $a = 1$ and use the following Galerkin orthogonality:

$$\int_0^1 (u-U)'v'\,dx = 0, \qquad \forall v \in V_h, \tag{9}$$

with v replaced by U to get

$$\begin{aligned}\|(u-U)'\|^2 &= \int_0^1 (u-U)'(u-U)'\,dx = \int_0^1 (u-U)'(u+U)'\,dx \\ &= \int_0^1 (u')^2\,dx - \int_0^1 (U')^2\,dx = -2F(u) + 2F(U),\end{aligned} \tag{10}$$

where we have used the identities

$$2F(u) = \|u'\|^2 - 2\int_0^1 fu\,dx = \{\text{with } w = u \text{ and } a = 1 \text{ in } (1)\} = -\|u'\|^2, \tag{11}$$

and similarly $2F(U) = -\|U'\|^2$.
(c) Recall that in the one dimensional case, we have the interpolation estimate, (see problem 1),

$$\|u' - U'\| \le C_i\|hf\|,$$

where C_i is an interpolation constant. This gives using (b) that

$$|F(U) - F(u)| \le C_i^2\|hf\|^2.$$

4. a) There is smooth function ϕ such that $\Delta\phi = 1$ so that, using Greens formula

$$\begin{aligned}\|u\|_\Omega^2 &= \int_\Omega u^2\,\Delta\phi = \int_{\partial\Omega} u^2\,\partial_n\phi - \int_\Omega 2u\nabla u\cdot\nabla\phi \\ &\le C_1\|u\|_{\partial\Omega}^2 + C_2\|u\|\|\nabla u\| \le C_1\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|u\|_\Omega^2 + \frac{1}{2}C_2^2\|\nabla u\|_\Omega^2.\end{aligned}$$

This yields

$$\|u\|_\Omega^2 \le 2C_1\|u\|_{\partial\Omega}^2 + C_2^2\|\nabla u\|_\Omega^2 \le C^2(\|u\|_{\partial\Omega}^2 + \|\nabla u\|_\Omega^2),$$

where $C^2 = \max(2C_1, C_2^2)$, $C_1 = \max_{\partial\Omega}|\partial_n\phi|$, and $C_2 = \max_\Omega(2|\nabla\phi|)$.
b) Multiply the equation $-\Delta u = f$ by u and integrate over Ω. Partial integration together with the boundary data $-\partial_{\mathbf{n}}u = ku$ and Cauchy's inequality, yields

$$\begin{aligned}\|\nabla u\|_\Omega^2 + k\|u\|_{\partial\Omega}^2 &= \int_\Omega \nabla u\cdot\nabla u + \int_{\partial\Omega} u(-\partial_n u) = \int_\Omega u(-\Delta u) = \int_\Omega fu \\ &\le \|u\|_\Omega^{\|}f\|_\Omega \le C_\Omega(\|u\|_{\partial\Omega} + \|\nabla u\|_\Omega)\|f\|_\Omega = \|u\|_{\partial\Omega}C_\Omega\|f\|_\Omega + \|\nabla u\|_\Omega C_\Omega\|f\|_\Omega \\ &\le \frac{1}{2}\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|\nabla u\|_\Omega^2 + C_\Omega^2\|f\|_\Omega^2.\end{aligned}$$

Subtracting $\frac{1}{2}\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|\nabla u\|_\Omega^2$ from the both sides, we end up with

$$(k - \frac{1}{2})\|u\|_{\partial\Omega}^2 \le \frac{1}{2}\|\nabla u\|_\Omega^2 + (k - \frac{1}{2})\|u\|_{\partial\Omega}^2 \le C_\Omega^2\|f\|_\Omega^2,$$

which gives that $\|u\|_{\partial\Omega} \to 0$ as $k \to \infty$.

5. See the Book and/or Lecture Notes.

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