Mathematic Chalmers & GU

**TMA372/MMG800: Partial Differential Equations, 2013–06–01, 8:30-12:30 V Halls**

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p- For GU students **G**:15-24p, **VG**: 25p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1213/index.html

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**1.** $\pi_1 f$ is the linear interpolant of a twice continuously differentiable function $f$ on $I$. Prove that

$$||f - \pi_1 f||_{L_1(I)} \le (b-a)^2 ||f''||_{L_1(I)}, \qquad I = (a,b).$$

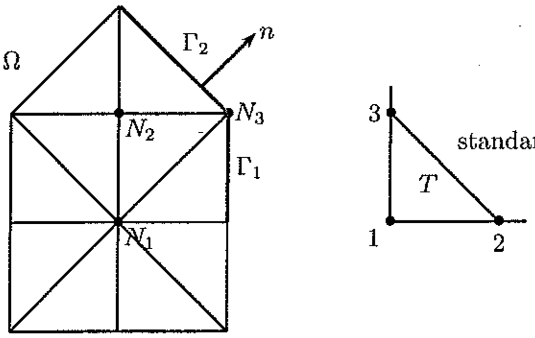
**2.** Derive $cG(1)$ *a priori* and *a posteriori* error estimates, in the norm $||e_x||$ for the problem,

$$-u_{xx} + u_x = f, \quad x \in (0,1); \quad u(0) = u(1) = 0. \qquad e := u - u_h$$

**3.** Formulate the cG(1) piecewise continuous Galerkin method for the boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2), \quad \nabla u \cdot n = 0, \quad x \in \Gamma_1 \cup \Gamma_2,$$

on the domain $\Omega$, with outward unit normal $n$ at the boundary (see fig.). Write the matrices for the resulting equation system using the following mesh with nodes at $N_1$, $N_2$ and $N_3$.




**4.** a) Show that the $L_2$ norm of the solution to the following *Śchrödinger* equation is time independent

$$\dot{u} + i\Delta u = 0, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \qquad i = \sqrt{-1}, \quad u = u_1 + iu_2.$$

Hint: Multiply the equation by $\bar{u} = u_1 - iu_2$, integrate over $\Omega$ and consider the real part.
b) Consider the corresponding *eigenvalue problem*, of finding $(\lambda, u \neq 0)$, such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

Show that $\lambda > 0$, and give the relation between $||u||$ and $||\nabla u||$ for $\lambda$:s eigenfunction $u$.
c) What is the optimal constant $C$ (expressed in terms of smallest eigenvalue $\lambda_1$), for which the inequality $||u|| \le C||\nabla u||$ can fullfil for all functions $u$, such that $u = 0$ on $\partial\Omega$?

**5.** Formulate and prove the Lax-Milgram Theorem

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**1.** Let $\lambda_0(x)=\frac{\xi_1-x}{\xi_1-x_0}$ and $\lambda_1(x)=\frac{x-\xi_0}{\xi_1-x_0}$ be two linear base functions. Then by the integral form of the Taylor formula we may write

$$\begin{cases} f(\xi_0) &= f(x)+f'(x)(\xi_0-x)+\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy,\\ f(\xi_1) &= f(x)+f'(x)(\xi_1-x)+\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy,\end{cases}$$

Therefore

$$\begin{aligned}\Pi_1 f(x) &= f(\xi_0)\lambda_0(x)+f(\xi_1)\lambda_1(x)\\ &= f(x)+\lambda_0(x)\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy+\lambda_1(x)\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy\end{aligned}$$

and by the triangle inequality we get

$$\begin{aligned}|f(x)-\Pi_1 f(x)| &= \Big|\lambda_0(x)\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy+\lambda_1(x)\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy\Big|\\ &\le |\lambda_0(x)|\Big|\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy\Big|+|\lambda_1(x)|\Big|\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy\Big|\\ &\le |\lambda_0(x)|\int_x^{\xi_0}|\xi_0-y||f''(y)|\,dy+|\lambda_1(x)|\int_x^{\xi_1}|\xi_1-y||f''(y)|\,dy\\ &\le |\lambda_0(x)|\int_x^{\xi_0}(b-a)|f''(y)|\,dy+|\lambda_1(x)|\int_x^{\xi_1}(b-a)|f''(y)|\,dy\\ &\le (b-a)\Big(|\lambda_0(x)|+|\lambda_1(x)|\Big)\int_a^b|f''(y)|\,dy\\ &= (b-a)\Big(\lambda_0(x)+\lambda_1(x)\Big)\int_a^b|f''(y)|\,dy=(b-a)\int_a^b|f''(y)|\,dy.\end{aligned}$$

Consequently

$$\int_a^b|f(x)-\Pi_1 f(x)|dx\le\int_a^b(b-a)\Big(\int_a^b|f''(y)|\,dy\Big)\,dx,=(b-a)^2\|f''\|_{L_1(I)}.$$

**2.** We multiply the differential equation by a test function $v\in H_0^1=\{v:\|v\|+\|v'\|<\infty,\ v(0)=v(1)=0\}$ and integrate over $I$. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u\in H_0^1(I)$ such that

$$\text{(1)}\qquad \int_I(u'v'+u'v)=\int_I fv,\quad\forall v\in H_0^1(I).$$

Or equivalently, find $u\in H_0^1(I)$ such that

$$\text{(2)}\qquad (u_x,v_x)+(u_x,v)=(f,v),\quad\forall v\in H_0^1(I),$$

with $(\cdot,\cdot)$ denoting the $L_2(I)$ scalar product: $(u,v)=\int_I u(x)v(x)\,dx$. A *Finite Element Method* with $cG(1)$ reads as follows: Find $u_h\in V_h^0$ such that

$$\text{(3)}\qquad \int_I(u_h'v'+u_h'v)=\int_I fv,\quad\forall v\in V_h^0\subset H_0^1(I),$$

where

$$V_h^0=\{v: v \text{ is piecewise linear and continuous in a partition of } I,\ v(0)=v(1)=0\}.$$

Or equivalently, find $u_h \in V_h^0$ such that

(4) $$(u_{h,x}, v_x) + (u_{h,x}, v) = (f, v), \quad \forall v \in V_h^0.$$

Let now

$$a(u,v) = (u_x, v_x) + (u_x, v).$$

We want to show that $a(\cdot,\cdot)$ is both elliptic and continuous:

*ellipticity*

(5) $$a(u,u) = (u_x, u_x) + (u_x, u) = \|u_x\|^2,$$

where we have used the boundary data, viz,

$$\int_0^1 u_x u\,dx = \left[\frac{u^2}{2}\right]_0^1 = 0.$$

*continuity*

(6) $$a(u,v) = (u_x, v_x) + (u_x, v) \le \|u_x\|\|v_x\| + \|u_x\|\|v\| \le 2\|u_x\|\|v_x\|,$$

where we used the Poincare inequality $\|v\| \le \|v_x\|$.

Let now $e = u - u_h$, then (2)- (4) gives that

(7) $$a(u - u_h, v) = (u_x - u_{h,x}, v_x) + (u_x - u_{h,x}, v) = 0, \quad \forall v \in V_h^0, \text{(Galerkin Orthogonality)}.$$

*A priori error estimate:* We use ellipticity (5), Galerkin orthogonality (7), and the continuity (6) to get

$$\|u_x - u_{h,x}\|^2 = a(u - u_h, u - u_h) = a(u - u_h, u - v) \le 2\|u_x - u_{h,x}\|\|u_x - v_x\|, \quad \forall v \in V_h^0.$$

This gives that

(8) $$\|u_x - u_{h,x}\| \le 2\|u_x - v_x\|, \quad \forall v \in V_h^0,$$

If we choose $v = \pi_h u \in V_h^0$, the interpolant of $u$, and use the interpolation estimate we get from (8) that

(9) $$\|u_x - u_{h,x}\| \le 2\|u_x - (\pi u)_x\| \le 2C_i\|h u_{xx}\|.$$

*A posteriori error estimate:* We use again ellipticity (5), Galerkin orthogonality (7), and the variational formulation (1) to get

(10) $$\begin{aligned}\|e_x\|^2 &= a(e,e) = a(e, e - \pi e) = a(u, e - \pi e) - a(u_h, e - \pi e)\\ &= (f, e - \pi e) - a(u_h, e - \pi e) = (f, e - \pi e) - (u_{h,x}, e_x - (\pi e)_x) - (u_{h,x}, e - \pi e)\\ &= (f - u_{h,x}, e - \pi e) \le C\|h(f - u_{h,x})\|\|e_x\|,\end{aligned}$$

where in the last equality we use the fact that $e(x_j) = (\pi e)(x_j)$, for $j$:s being the node points, also $u_{h,xx} \equiv 0$ on each $I_j := (x_{j-1}, x_j)$. Thus

$$(u_{h,x}, e_x - (\pi e)_x) = -\sum_j \int_{I_j} u_{h,xx}(e - \pi e) + \sum_j \Big(u_{h,x}(e - \pi e)\Big)\Big|_{I_j} = 0.$$

Hence, (10) yields:

(11) $$\|e_x\| \le C\|h(f - u_{h,x})\|.$$

**3.** Let $V$ be the linear function space defined by

$$V_h := \{v : v \text{ is continuous in } \Omega,\ v = 0, \text{ on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)\}.$$

Multiplying the differential equation by $v \in V$ and integrating over $\Omega$ we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$\begin{aligned}
-(\Delta u, \nabla v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v\, ds \\
&= (\nabla u, \nabla v) - \int_{\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)} (n \cdot \nabla u) v\, ds - \int_{\Gamma_1 \cup \Gamma_2} (n \cdot \nabla u) v\, ds \\
&= (\nabla u, \nabla v), \qquad \forall v \in V.
\end{aligned}$$

Thus the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let $V_h$ be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$: The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{i=1}^{3} \xi_i \varphi_i(x)$, where $\varphi_i$ are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^{3} \xi_i \Big( \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\, dx + \int_\Omega \varphi_i \varphi_j\, dx \Big) = \int_\Omega f \varphi_j\, dx, \quad j = 1, 2, 3,$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness matrix for the reference triangle $T$. The local basis functions are

$$\begin{aligned}
\phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$\begin{aligned}
m_{11} &= (\phi_1, \phi_1) = \int_T \phi_1^2\, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2\, dx_1 dx_2 = \frac{h^2}{12}, \\
s_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\, dx = \frac{2}{h^2}|T| = 1.
\end{aligned}$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2\, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices $M$ and $S$ from the local ones $m$ and $s$:

$$\begin{aligned}
M_{11} &= 8m_{22} = \frac{8}{12}h^2, & S_{11} &= 8s_{22} = 4,\\
M_{12} &= 2m_{12} = \frac{1}{12}h^2, & S_{12} &= 2s_{12} = -1,\\
M_{13} &= 2m_{23} = \frac{1}{12}h^2, & S_{13} &= 2s_{23} = 0,\\
M_{22} &= 4m_{11} = \frac{4}{12}h^2, & S_{22} &= 4s_{11} = 4,\\
M_{23} &= 2m_{12} = \frac{1}{12}h^2,, & S_{23} &= 2s_{12} = -1,\\
M_{33} &= 3m_{22} = \frac{3}{12}h^2,, & S_{33} &= 3s_{22} = 3/2.
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3/2 \end{bmatrix}.$$

**4.** a) We multiply the shrödinger equation by $\bar{u}$ and integrate over $\Omega$ to obtain

$$\int_\Omega \bar{u}\dot{u} + i\int_\Omega \bar{u}\nabla u = \int_\Omega (u_1\dot{u}_1 + u_2\dot{u}_2) + i\int_\Omega (u_1\dot{u}_2 - u_2\dot{u}_1 - \nabla\bar{u}\cdot\nabla u) = 0.$$

Now both real and imaginary part of the above expression is 0. Thus, considering the real part, we have

$$\int_\Omega (u_1\dot{u}_1 + u_2\dot{u}_2) = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (u_1^2 + u_2^2) = 0,$$

therefore $\int_\Omega |u|^2$ is independent of the time.

b) Multiplying the eigenvalue equation $-\Delta u = \lambda u$ by $u$, integrating over $\Omega$, and using partial integration we get

$$\lambda\int_\Omega u^2 = \int_\Omega u(-\Delta u) = \int_\Omega |\nabla u|^2,$$

which gives $\lambda \geq 0$ (and also $\lambda > 0$, for $u \neq 0$). Further $||u|| = \frac{1}{\sqrt{\lambda}}||\nabla u||$. This indicates that the constant in the estimate $||u|| \leq C||\nabla u||$, satisfying for all functions $u$ with $u = 0$ on $\Gamma := \partial\Omega$, can not be smaller than $\frac{1}{\sqrt{\lambda_1}}$, with $\lambda_1 > 0$ being the smallest eigenvalue. As a matter of fact we have the inequality $||u|| \leq \frac{1}{\sqrt{\lambda_1}}||\nabla u||$, for all $u$ with $u = 0$ on $\Gamma$. This is due to the fact that we can represent $u$ in terms of orthogonal eigenfunctions both "with and without gradient", i.e. $\int_\Omega u_i u_j = \int_\Omega \nabla u_i \cdot \nabla u_j = 0$, for $i \neq j$.

**5.** See the Book and/or Lecture Notes.

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