Mathematic Chalmers & GU

**TMA372/MMG800: Partial Differential Equations, 2012–08–29, kl 8:30-12:30 V Halls**

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p-    For GU students**G**:15-24p, **VG**: 25p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1112/index.html

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**1.** Prove the following error estimate for the linear interpolation for a function $f \in C^2(0,1)$,

$$||\pi_1 f - f||_{L_\infty(0,1)} \leq \frac{1}{8} \max_{0\leq\xi\leq 1} |f''(\xi)|.$$

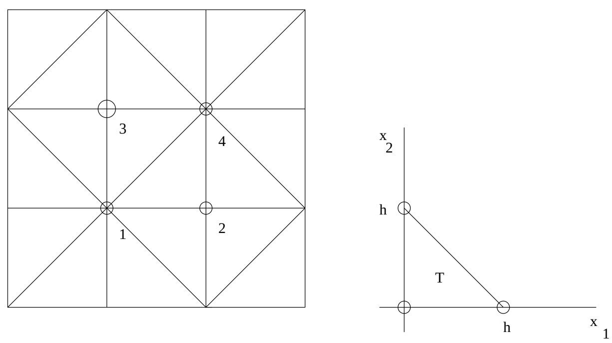
**2.** Let $\alpha$ and $\beta$ be positive constants. Give the piecewise linear finite element approximation procedure, on the uniform mesh, for the problem

$$-u''(x) = 1, \quad 0 < x < 1; \qquad u(0) = \alpha, \quad u'(1) = \beta.$$

**3.** Formulate the $cG(1)$ method for the boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega.$$

Write down the matrix form of the resulting equation system using the following uniform mesh:




**4.** Prove an a priori and an a posteriori error estimate for the cG(1) finite element method for

$$-u''(x) + xu'(x) + u(x) = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

in the energy norm $||v||_E$ with $||v||_E^2 = ||v||_{L_2(I)}^2 + ||v'||_{L_2(I)}^2$, $\qquad I := (0.1)$.

**5.** Formulate and prove the Lax-Milgram theorem.

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void!

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**1.** By the Lagrange interpolation theorem

$$||f-\pi_1 f||_{L_\infty(0,1)} \le \frac{1}{2}(x-0)\cdot(1-x)\max_{x\in[0,1]}|f''|.$$

Further, the function $g(x)=x(1-x)$ has minimum when $g'(x)=0$, i.e. $1\cdot(1-x)+x\cdot(-1)=0$, or for $x=1/2$. Therefore, $\max_{x\in[0,1]}[x(1-x)]=\max_{x\in[0,1]} g(x)=1/2(1-1/2)=1/4$. Hence

$$||f-\pi_1 f||_{L_\infty(0,1)} \le \frac{1}{8}||f||_{L_\infty(0,1)}.$$

**2.** Multiply the pde by a test function $v$ with $v(0)=0$, integrate over $x\in(0,1)$ and use partial integration to get

$$\begin{aligned}
&-[u'v]_0^1+\int_0^1 u'v'\,dx=\int_0^1 v\,dx &&\Longleftrightarrow\\
&-u'(1)v(1)+u'(0)v(0)+\int_0^1 u'v'\,dx=\int_0^1 v\,dx &&\Longleftrightarrow\\
&-\beta v(1)+\int_0^1 u'v'\,dx=\int_0^1 v\,dx.
\end{aligned} \tag{1}$$

The continuous variational formulation is now formulated as follows: Find

$$(VF)\qquad u\in V:=\{w:\int_0^1\Big(w(x)^2+w'(x)^2\Big)\,dx<\infty,\quad w(0)=\alpha\},$$

such that

$$\int_0^1 u'v'\,dx=\int_0^1 v\,dx+\beta v(1),\quad \forall v\in V^0,$$

where

$$V^0:=\{v:\int_0^1\Big(v(x)^2+v'(x)^2\Big)\,dx<\infty,\quad v(0)=0\}.$$

For the discrete version we let $\mathcal{T}_h$ be a uniform partition: $0=x_0<x_1<\ldots<x_{M+1}$ of $[0,1]$ into the subintervals $I_n=[x_{n-1},x_n],\ n=1,\ldots M+1$. Here, we have $M$ interior nodes: $x_1,\ldots x_M$, two boundary points: $x_0=0$ and $x_{M+1}=1$ and hence $M+1$ intervals.

The finite element method (discrete variational formulation) is now formulated as follows: Find

$$(FEM)\qquad U\in V_h:=\{w_h: w_h \text{ is piecewise linear, continuous on } \mathcal{T}_h,\ w_h(0)=\alpha\},$$

such that

$$\int_0^1 U'v_h'\,dx=\int_0^1 v_h\,dx+\beta v_h(1),\quad \forall v\in V_h^0, \tag{2}$$

where

$$V_h^0:=\{v_h: v_h \text{ is piecewise linear, continuous on } \mathcal{T}_h,\ v_h(0)=0\}.$$

Using the basis functions $\varphi_j,\ j=0,\ldots M+1$, where $\varphi_1,\ldots\varphi_M$ are the usual *hat-functions* whereas $\varphi_0$ and $\varphi_{M+1}$ are *semi-hat-functions* viz;

$$\varphi_j(x)=\begin{cases}0, & x\notin[x_{j-1},x_j]\\ \frac{x-x_{j-1}}{h} & x_{j-1}\le x\le x_j\\ \frac{x_{j+1}-x}{h} & x_j\le x\le x_{j+1}\end{cases},\qquad j=1,\ldots M. \tag{3}$$

and

$$\varphi_0(x) = \begin{cases} \frac{x_1 - x}{h} & 0 \le x \le x_1 \\ 0, & x_1 \le x \le 1 \end{cases}, \qquad \varphi_{M+1}(x) = \begin{cases} \frac{x - x_M}{h} & x_M \le x \le x_{M+1} \\ 0, & 0 \le x \le x_M. \end{cases}$$

In this way we may write

$$V_h = \alpha\varphi_0 \oplus [\varphi_1, \ldots, \varphi_{M+1}], \quad V_h^0 = [\varphi_1, \ldots, \varphi_{M+1}].$$

Thus every $U \in V_h$ can ve written as $U = \alpha\varphi_0 + v_h$ where $v_h \in V_h^0$, i.e.,

$$U = \alpha\varphi_0 + \xi_{1\varphi_1} + \ldots \xi_{M+1}\varphi_{M+1} = \alpha\varphi_0 + \sum_{i=1}^{M+1} \xi_i\varphi_i \equiv \alpha\varphi_0 + \tilde{U},$$

where $\tilde{U} \in V_h^0$, and hence the problem (2) can equivalently be formulated as to find $\xi_1, \ldots \xi_{M+1}$ such that

$$\int_0^1 \Big(\alpha\varphi_0' + \sum_{i=1}^{M+1} \xi_i\varphi_i'\Big)\varphi_j'\, dx = \int_0^1 \varphi_j\, dx + \beta\varphi_j(1), \quad j = 1, \ldots M+1,$$

which can be written as

$$\sum_{i=1}^{M+1} \Big(\int_0^1 \varphi_j'\varphi_i'\, dx\Big)\xi_i = -\int_0^1 \varphi_0'\varphi_j'\, dx + \int_0^1 \varphi_j\, dx + \beta\varphi_j(1), \quad j = 1, \ldots M+1,$$

or equivalently $A\xi = b$ where $A = (a_{ij})$ is the tridiagonal matrix with entries

$$a_{ii} = 2, \quad a_{i,i+1} = a_{i+1,i} = -1, \quad i = 1, \ldots M, \quad \text{and} \quad a_{M+1,M+1} = 1,$$

i.e.,

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 \\ \ldots & & & & & & \\ \ldots & & & & & & \\ \ldots & & & & & & \\ 0 & 0 & \ldots & 0 & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & 0 & -1 & 1 \end{bmatrix},$$

and the unkown $\xi$ and the data $b$ are given by

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \xi_M \\ \xi_{M+1} \end{bmatrix}, \qquad b = \begin{bmatrix} \int_0^1 \varphi_1\, dx - \alpha\int_0^1 \varphi_0'\varphi_1'\, dx \\ \int_0^1 \varphi_2\, dx \\ \cdot \\ \cdot \\ \int_0^1 \varphi_M\, dx \\ \int_0^1 \varphi_{M+1}\, dx + \beta\varphi_{M+1}(1) \end{bmatrix} = \begin{bmatrix} h + \frac{1}{h}\alpha \\ h \\ \cdot \\ \cdot \\ h \\ \frac{h}{2} + \beta \end{bmatrix}.$$

**3.** Let $V_h$ be the usual finite element space cosisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega$. The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{i=1}^4 \xi_i\varphi_i(x)$, where $\varphi_i$ are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^4 \xi_i\Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\, dx + \int_\Omega \varphi_i\varphi_j\, dx\Big) = \int_\Omega f\varphi_j\, dx, \quad j = 1, \ldots, 4,$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness amtrix for the reference triangle $T$. The local basis functions are

$$\phi_1(x_1,x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1,x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\phi_2(x_1,x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\phi_3(x_1,x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1,\phi_1) = \int_T \phi_1^2\,dx = h^2\int_0^1\int_0^{1-x_2}(1-x_1-x_2)^2\,dx_1dx_2 = \frac{h^2}{12},$$
$$s_{11} = (\nabla\phi_1,\nabla\phi_1) = \int_T |\nabla\phi_1|^2\,dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1,\phi_1) = \int_T \phi_1^2\,dx = \frac{|T|}{3}\sum_{j=1}^{3}\phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0+\frac{1}{4}+\frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoins of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices $M$ and $S$ from the local ones $m$ and $s$:

$$M_{11} = M_{44} = 8m_{22} = \frac{8}{12}h^2, \qquad S_{11} = S_{44} = 8s_{22} = 4,$$
$$M_{12} = M_{13} = M_{24} = M_{34} = 2m_{12} = \frac{1}{12}h^2, \quad S_{12} = S_{13} = S_{24} = S_{34} = 2s_{12} = -1,$$
$$M_{14} = 2m_{23} = \frac{1}{12}h^2, \qquad S_{14} = 2s_{23} = 0,$$
$$M_{22} = M_{33} = 4m_{11} = \frac{4}{12}h^2, \qquad S_{22} = S_{33} = 4s_{11} = 4,$$
$$M_{23} = 0, \qquad S_{23} = 0.$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

**4.** We multiply the differential equation by a test function $v \in H_0^1 = \{v : ||v|| + ||v'|| < \infty,\ v(0) = 0\}$ and integrate over $I$. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

(4) $$\int_I (u'v' + xu'v + uv) = \int_I fv, \quad \forall v \in H_0^1(I).$$

A *Finite Element Method* with *cG*(1) reads as follows: Find $U \in V_h^0$ such that

(5) $$\int_I (U'v' + xU'v + Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I),$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (4)-(5) gives that

$$\int_I (e'v' + xe'v + ev) = 0, \quad \forall v \in V_h^0, \qquad \text{(Galerkin Ortogonalitet)}. \tag{6}$$

We note that using $e(0) = e(1) = 0$, we get

$$\int_I xe'e = \frac{1}{2}\int_I x\frac{d}{dx}(e^2) = \frac{1}{2}(xe^2)|_0^1 - \frac{1}{2}\int_I e^2 = -\frac{1}{2}\int_I e^2, \tag{7}$$

Further, using Poincare inequality we have

$$\|e\|^2 \le \|e'\|^2.$$

*A priori error estimate:* We use (6) and (7) to get

$$\begin{aligned}
\|e'\|^2_{L_2(I)} + \frac{1}{2}\|e\|^2_{L_2} &= \int_I (e'e' + \frac{1}{2}ee) = \int_I (e'e' + xe'e + ee)\\
&= \int_I \Big(e'(u-U)' + xe'(u-U) + e(u-U)\Big) = \{v = U - \pi_h u \text{ i(6)}\}\\
&= \int_I \Big(e'(u-\pi_h u)' + xe'(u-\pi_h u) + e(u-\pi_h u)\Big)\\
&\le \|(u-\pi_h u)'\|\|e'\| + \|u-\pi_h u\|\|e'\| + \|u-\pi_h u\|\|e\|\\
&\le \{\|(u-\pi_h u)'\| + \sqrt{2}\|u-\pi_h u\|\}\|e\|_{H^1}\\
&\le C_i\{\|hu''\| + \sqrt{2}\|h^2u''\|\}\|e\|_{H^1}.
\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le 2C_i\{\|hu''\| + \sqrt{2}\|h^2u''\|\}.$$

which is the a priori error estimate.

*A posteriori error estimate:*

$$\begin{aligned}
\|e'\|^2_{L_2(I)} + \frac{1}{2}\|e\|^2_{L_2} &= \int_I (e'e' + \frac{1}{2}ee) = \int_I (e'e' + xe'e + ee)\\
&= \int_I ((u-U)'e' + x(u-U)'e + (u-U)e) = \{v = e \ \text{ in } (4)\}\\
&= \int_I fe - \int_I (U'e' + xU'e + Ue) = \{v = \pi_h e \ \text{ in } (6)\} \qquad (8)\\
&= \int_I f(e-\pi_h e) - \int_I \Big(U'(e-\pi_h e)' + xU'(e-\pi_h e) + U(e-\pi_h e)\Big)\\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e-\pi_h e),
\end{aligned}$$

where $\mathcal{R}(U) := f + U'' - xU' - U = f - xU' - U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$\|e'\|^2_{L_2(I)} + \frac{1}{2}\|e\|^2_{L_2} \le \|h\mathcal{R}(U)\|\|h^{-1}(e-\pi_h e)\| \le C_i\|h\mathcal{R}(U)\|\|e'\| \le \frac{1}{2}C_i^2\|h\mathcal{R}(U)\|^2 + \frac{1}{2}\|e'\|^2_{L_2(I)},$$

where $C_i$ is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

**5.** See the Book and/or Lecture Notes.

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