Mathematic Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2012–06–02, kl 8:30-12:30 V Halls

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p- For GU students**G**:15-24p, **VG**: 25p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1112/index.html

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1. Consider the initial value problem: $\dot{u}(t) + au(t) = 0, \quad t > 0, \quad u(0) = u_0, \quad a > 0,$ (constant). Assume a constant time step k and verify the iterative formulas for $dG(0)$ and $dG(1)$ approximations U and $\tilde{U}$ respectively: i.e.

$$U_n = \Big(\frac{1}{1+ak}\Big)^n u_0, \qquad \tilde{U}_n = \Big(\frac{1-ak/2}{1+ak/2}\Big)^n u_0.$$

2. Prove an a priori and an a posteriori error estimate for the cG(1) finite element method for

$$-u''(x) + xu'(x) + u(x) = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

in the energy norm $||v||_E$ with $||v||_E^2 = ||v||_{L_2(I)}^2 + ||v'||_{L_2(I)}^2$.

3. Consider the Dirichlet boundary value problem

$$-\nabla \cdot (a(x)\nabla u) = f(x), \quad x \in \Omega \subset \mathbb{R}^2, \qquad u = 0, \text{ for } x \in \partial\Omega.$$

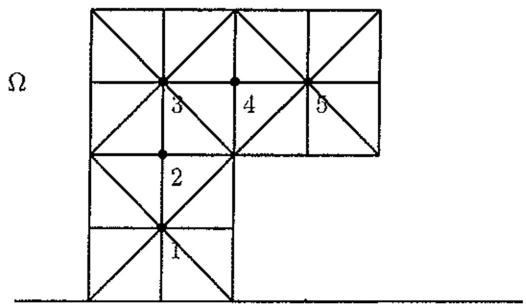
Assume that c_0 and c_1 are constants such that $c_0 \le a(x) \le c_1$, $\forall x \in \Omega$ and let $U = \sum_{j=1}^N \alpha_j w_j(x)$ be a Galerkin approximation of u in a finite dimensional subspace M of $H_0^1(\Omega)$. Prove the a priori error estimate below and specify C as best you can

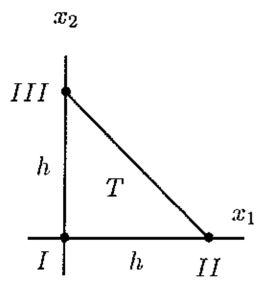
$$||u - U||_{H_0^1(\Omega)} \le C \inf_{\chi \in M} ||u - \chi||_{H_0^1(\Omega)}.$$

4. Formulate the cG(1) Galerkin finite element method for the Dirichlet boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega,$$

on a smooth domain Ω. Write the matrices for the resulting equation system using the partition below (see fig.) with the nodes at N_1, N_2, N_3, N_4 and N_5 and a uniform mesh size h.
Hint: You may first compute the matrices for the reference triangle-element T.





5. Prove the Poincare' inequality in a convex domain $\Omega \subset R^2$: There is a constant C depending in Ω such that for all $v \in H_0^1(\Omega)$, (all L^2 functions v with $\nabla v \in L^2$ and $v = 0$ at the boundary $\partial\Omega$):

$$||v||_{L^2(\Omega)}^2 \le C||\nabla v||_{L^2(\Omega)}^2, \qquad \text{where } ||w||_{L^2(\Omega)} = \Big(\int_\Omega |w|^2\,dx\Big)^{1/2}.$$

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1. See the Book and/or Lecture Notes.

2. We multiply the differential equation by a test function $v \in H_0^1 = \{v : \|v\| + \|v'\| < \infty,\ v(0) = 0\}$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + xu'v) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{1}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$\int_I (U'v' + xU'v + Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{2}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I (e'v' + xe'v + ev) = 0, \quad \forall v \in V_h^0, \qquad \text{(Galerkin Ortogonalitet)}. \tag{3}$$

We note that using $e(0) = e(1) = 0$, we get

$$\int_I xe'e = \frac{1}{2}\int_I x\frac{d}{dx}(e^2) = \frac{1}{2}(xe^2)|_0^1 - \frac{1}{2}\int_I e^2 = -\frac{1}{2}\int_I e^2, \tag{4}$$

Further, using Poincare inequality we have

$$\|e\|^2 \leq \|e'\|^2.$$

A priori error estimate: We use (3) and (4) to get

$$\begin{aligned}
\|e'\|^2_{L_2(I)} + \frac{1}{2}\|e\|^2_{L_2} &= \int_I (e'e' + \frac{1}{2}ee) = \int_I (e'e' + xe'e + ee) \\
&= \int_I \Big(e'(u-U)' + xe'(u-U) + e(u-U)\Big) = \{v = U - \pi_h u \text{ i(3)}\} \\
&= \int_I \Big(e'(u-\pi_h u)' + xe'(u-\pi_h u) + e(u-\pi_h u)\Big) \\
&\leq \|(u-\pi_h u)'\|\|e'\| + \|u-\pi_h u\|\|e'\| + \|u-\pi_h u\|\|e\| \\
&\leq \{\|(u-\pi_h u)'\| + \sqrt{2}\|u-\pi_h u\|\}\|e\|_{H^1} \\
&\leq C_i\{\|hu''\| + \sqrt{2}\|h^2u''\|\}\|e\|_{H^1}.
\end{aligned}$$

this gives that

$$\|e\|_{H^1} \leq 2C_i\{\|hu''\| + \sqrt{2}\|h^2u''\|\}.$$

which is the a priori error estimate.

A posteriori error estimate:

$$
\begin{aligned}
\|e'\|^2_{L_2(I)} + \frac{1}{2}\|e\|^2_{L_2} &= \int_I (e'e' + \frac{1}{2}ee) = \int_I (e'e' + xe'e + ee) \\
&= \int_I ((u-U)'e' + x(u-U)'e + (u-U)e) = \{v = e \ \text{ in } (1)\} \\
&= \int_I fe - \int_I (U'e' + xU'e + Ue) = \{v = \pi_h e \ \text{ in } (3)\} \qquad (5)\\
&= \int_I f(e-\pi_h e) - \int_I \Big(U'(e-\pi_h e)' + xU'(e-\pi_h e) + U(e-\pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e-\pi_h e),
\end{aligned}
$$

where $\mathcal{R}(U) := f + U'' - xU' - U = f - xU' - U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$\|e'\|^2_{L_2(I)} + \frac{1}{2}\|e\|^2_{L_2} \le \|h\mathcal{R}(U)\|\,\|h^{-1}(e-\pi_h e)\| \le C_i\|h\mathcal{R}(U)\|\,\|e'\| \le \frac{1}{2}C_i^2\|h\mathcal{R}(U)\|^2 + \frac{1}{2}\|e'\|^2_{L_2(I)},$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

3. Recall the continuous and approximate weak formulations:

$$(a\nabla u, \nabla v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{6}$$

and

$$(a\nabla U, \nabla v) = (f, v), \qquad \forall v \in M, \tag{7}$$

respectively, so that

$$(a\nabla(u-U), \nabla v) = 0, \qquad \forall v \in M. \tag{8}$$

We may write

$$u - U = u - \chi + \chi - U,$$

where χ is an arbitrary element of M, it follows that

$$
\begin{aligned}
(a\nabla(u-U), \nabla(u-U)) &= (a\nabla(u-U), \nabla(u-\chi)) \\
&\le ||a\nabla(u-U)|| \cdot ||u-\chi||_{H_0^1(\Omega)} \qquad (9)\\
&\le c_1||u-U||_{H_0^1(\Omega)}||u-\chi||_{H_0^1(\Omega)},
\end{aligned}
$$

on using (3), Schwarz's inequality and the boundedness of a. Also, from the boundedness condition on a, we have that

$$(a\nabla(u-U), \nabla(u-U)) \ge c_0||u-U||^2_{H_0^1(\Omega)}. \tag{10}$$

Combining (4) and (5) gives

$$||u-U||_{H_0^1(\Omega)} \le \frac{c_1}{c_0}||u-\chi||_{H_0^1(\Omega)}.$$

Since χ is an arbitrary element of M, we obtain the result.

4. Let V be the linear function space defined by

$$V := \{v : v \text{ is continuous in } \Omega,\ v = 0, \text{ on } \partial\Omega\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$-(\Delta u, \nabla v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds = (\nabla u, \nabla v), \qquad \forall v \in V.$$

Thus, since $v = 0$ on $\partial\Omega$, the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let now V_h be the usual finite element space consisting of continuous piecewise linear functions, on the given partition (triangulation), satisfying the boundary condition $v = 0$ on $\partial\Omega$:

$$V_h := \{v : v \text{ is continuous piecewise linear in } \Omega,\ v = 0, \text{ on } \partial\Omega\}.$$

The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^{5} \xi_i \varphi_j(x)$, where φ_j are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{5} \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \varphi_j \, dx \Big) = \int_\Omega f \varphi_i \, dx, \quad i = 1,2,3,4,5$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1, x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1, x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\phi_2(x_1, x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\phi_3(x_1, x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 dx_2 = \frac{h^2}{12},$$

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$M_{11} = M_{33} = M_{55} = 8m_{22} = 8 \times \frac{h^2}{12}, \qquad S_{11} = S_{33} = S_{55} = 8s_{22} = 8 \times \frac{1}{2}8 = 4,$$

$$M_{22} = M_{44} = 4m_{11} = 4 \times \frac{h^2}{12} = \frac{h^2}{3}, \qquad S_{22} = S_{44} = 4s_{11} = 4 \times 1 = 4,$$

$$M_{12} = M_{23} = M_{34} = M_{45} = 2m_{12} = \frac{1}{12}h^2, \qquad S_{12} = S_{23} = S_{34} = S_{45} = 2s_{12} = -1,$$

$$M_{13} = M_{14} = M_{15} = M_{24} = M_{25} = M_{35} = 0, \qquad S_{13} = S_{14} = S_{15} = S_{24} = S_{25} = S_{35} = 0,$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 8 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}.$$

5. See the Book and/or Lecture Notes.

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