Mathematic Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2009–08–26; kl 8.30-13.30.

Telephone: Karin Kraft: 0762-721861
Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 8p. Valid bonus poits will be added to the scores.
Breakings: **3**: 20-29p, **4**: 30-39p och **5**: 40p- GU **G**:20-39p, **VG**: 40p-

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1. Consider the boundary value problem:

$$(1)\qquad \begin{cases} -(a(x)u'(x))' = f(x), & \text{for} \quad 0 < x < 1, \\ u(0) = 0, \qquad a(1)u'(1) = g_1, & \end{cases}$$

Formulate a finite element method for this problem and show the *a posteriori error estimate*:

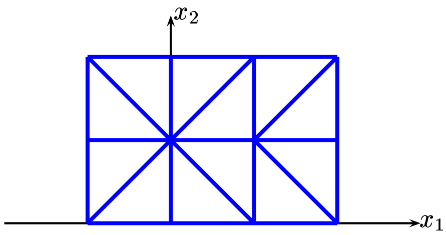
$$(2)\qquad ||(u-U)'||_a \leq C_i ||hR(U)||_{1/a}.$$

2. a) Let $a(x) = 1$ for $x < 1/2$, $a(x) = 2$ for $x \geq 1/2$ and $g_1 = -2$. Formulate a finite element method for (1). Dervie the matrix equation $AU = b$ arising in discretizing the problem by cG(1) FEM in a uniform partition of $I = (0,1)$ into 4 intervals. Compute, explicitely, only the matrix elements a_{22} and a_{34} and the vector element b_4.
b) Show, using (2), that the above finite element approximation is actually exact, i.e., $u - U = 0$.

3. Consider the problem

$$(3)\qquad \begin{cases} -\Delta u = f, & \text{in } \Omega = \{(x_1, x_2) : -1 < x_1 < 2,\ 0 < x_2 < 2\} \\ u = 0, & \text{0n } \Gamma = \partial\Omega, \end{cases}$$

where $f = 1$ for $x_1 < 0$ and $f = 2$ for $x_1 > 0$.
a) Write down the discrete system $SU = b$ (S is the stiffness matrix and b is the load vextor) in approximating (3) using cG(1) FEM in the following triangulation:



b) Consider the same problem as in a), replacing the Dirichlet $u = 0$ (only) on $x_1 = 2$ by the Neuman data: $\partial_n u = 0$ on $x_1 = 2$, $0 < x_2 < 2$.

4. Let $M \in (0,1)$. Consider the problem

$$(4)\qquad (1 - M^2)u_{xx} + u_{yy} = f, \qquad (x,y) \in \mathbb{R}^2.$$

Determine the solution u for $f(x) = g(x)\delta(x)$ where δ is the Dirac δ function and

$$g(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

Hint: The fundamental solution for $-\Delta$ in $\mathbb{R}^2$ is given by $E(x,y) = \frac{1}{2\pi} \log \frac{1}{\sqrt{x^2+y^2}}$.

5. Formulate and prove the Lax-Milgram theorem.

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Lösningar/Solutions.

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1. Define the continuous and discrete function spaces

$$V = \{v : \int_0^1 [(v')^2 + v^2]\, dx < \infty, \quad v(0) = 0\},$$

and

$$V_h = \{v \in V : v \text{is piecewise linear and continuous on the partition of} I = [0, 1]\},$$

Multiply the differential equation by a test function $v \in V$ and integrate over I. Partial integration yields the *variational formulation*: Find $u \in V$ such that

$$\int_0^1 au'v'\, dx = g_1 v(1), \qquad \forall v \in V.$$

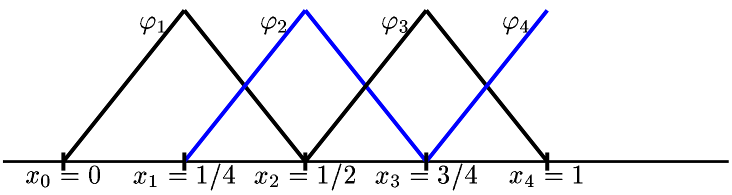
The corresponding *finite element method* is: Find $U \in V_h$ such that

$$\int_0^1 aU'v'\, dx = g_1 v(1), \qquad \forall v \in V_h.$$

For the inequality (2), se lecture notes.

2. a) A uniform partition for $I = (0, 1)$ into 4 subintervals $I_1 = (0, 1/4)$, $I_2 = (1/4, 1/2)$, $I_3 = (1/2, 3/4)$ and $I_4 = (3/4, 1)$ would have the piecewise linear basis functions $\{\varphi_j\}_{j=1}^4$, where for V_h defined by $\varphi_j \in V_h$ and

$$\varphi_j(x_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$



With the ansatz $U(x) = \sum_{j=1}^4 U_j \varphi_j(x)$ the FEM, in the basis functions $\{\varphi_j\}_{j=1}^4$, can be formulated as follows: Find $U \in V_h$ such that

$$\int_0^1 aU'\varphi_i'\, dx = g_1 \varphi_i(1), \qquad i = 1, 2, 3, 4.$$

Inserting the ansatz for U yields

$$\sum_{j=1}^4 U_j \int_0^1 a\varphi_j'\varphi_i'\, dx = g_1 \varphi_i(1), \qquad i = 1, 2, 3, 4.$$

In this way we obtain a matrix problem $AU = b$ with the element of A giver by

$$a_{ij} = \int_0^1 a\varphi_j'\varphi_i'\, dx, \qquad \text{and} \quad b_i = g_1 \varphi_i(1).$$

With $a(x)$ and g_1 given as in the problem and $h = 1/4$ we have that

$$a_{22} = \int_{1/4}^{1/2} 1 \cdot \frac{1}{h} \cdot \frac{1}{h}\, dx + \int_{1/2}^{3/4} 2 \cdot \Big(\frac{-1}{h}\Big) \cdot \Big(\frac{-1}{h}\Big)\, dx = 12,$$

and

$$a_{34} = \int_{3/4}^{1} 2 \cdot \frac{1}{h} \cdot \Big(\frac{-1}{h}\Big)\, dx = -8, \qquad \text{and} \qquad b_4 = (-2)\cdot 1 = -2.$$

b) Due to the fact tha $a(x)$ is chosen to br piecewise constant and $U \in V_h$ we get $R(U) = a'U' + aU'' = 0$. Thus by (2)

$$||e'||_a \le 0 \Longrightarrow e(x) = C.$$

Now since e is continuous and $e(0) = 0$, hence $e(x) = 0$.

3. Let V be the linear function space defined by

$$V_h := \{v : v \text{ is continuous in } \Omega,\ v = 0, \text{ on } \partial\Omega\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$-(\Delta u, \nabla v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v\, ds = (\nabla u, \nabla v), \qquad \forall v \in V.$$

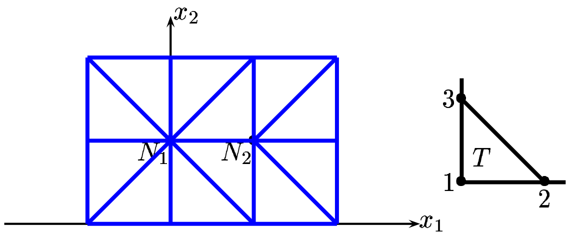
Thus the variational formulation is:

$$(\nabla u, \nabla v) = (f, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega$: The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) = (f, v) \qquad \forall v \in V_h$$

With this boundary conditions we have the internal nodes N_1 and N_2. Making the "Ansatz"



$U(x) = \sum_{j=1}^{2} \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^{2} \xi_j \int_{\Omega} \nabla\varphi_i \cdot \nabla\varphi_j\, dx = \int_{\Omega} f\varphi_j\, dx, \quad i = 1, 2,$$

or, in matrix form,

$$S\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix and $F_j = (f, \varphi_j)$ is the load vector. We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1, x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1, x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\phi_2(x_1, x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\phi_3(x_1, x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\, dx = \frac{2}{h^2}|T| = 1.$$

Similarly we can compute the other elements and obtain

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrix S from the local one s:

$$S_{11} = 8s_{22} = 4, \qquad S_{12} = 2s_{12} = -1,$$
$$S_{21} = 2s_{12} = -1, \qquad S_{22} = 2s_{11} + 4s_{22} = 2 + 2 = 4$$

As for the load vector we have

$$\int_\Omega f\varphi_1 = \int_{x_1<0} \varphi_1 + 2\int_{x_1>0} \varphi_1 = 4\cdot\frac{1}{3}\cdot\frac{1}{2} + 2\cdot 4\cdot\frac{1}{3}\cdot\frac{1}{2} = 2/3 + 4/3 = 2.$$

$$\int_\Omega f\varphi_2 = 2\int_{x_1>0} \varphi_2 = 2\cdot 6\cdot\frac{1}{3}\cdot\frac{1}{2} = 2$$

Thus the equatuion system is given by

$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

b) With the Neumann boundary data we obtain an addition node at $N_3 = (2,1)$ with the obvious corresponding basis function φ_3 which gives rize to an addtional row and an addtional column viz,

$$\int_\Omega \nabla\varphi_3\cdot\nabla\varphi_3 = 2, \quad \int_\Omega \nabla\varphi_2\cdot\nabla\varphi_3 = \int_\Omega \nabla\varphi_3\cdot\nabla\varphi_2 = -1 \quad \int_\Omega f\varphi_3 = 2\cdot\frac{1}{2}.$$

Consequently the corresponding system reads as

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2/3 \end{bmatrix}.$$

4. Notre that $M < 1$. The substitution of variables

$$\begin{cases} x_1' = \frac{1}{\sqrt{1-M^2}}x_1 \\ x_2' = x_2 \end{cases} \Longrightarrow \begin{cases} \frac{\partial}{\partial x_1} = \frac{1}{\sqrt{1-M^2}}\frac{\partial}{\partial x_1'}, & \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2'} \\ \frac{\partial^2}{\partial x_1^2} = \frac{1}{1-M^2}\frac{\partial^2}{\partial x_1'^2}, & \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial x_2'^2} \end{cases} \Longrightarrow u_{x_1'x_1'} + u_{x_2'x_2'} = f(x_1', x_2').$$

Here $f(x_1', x_2') = g(x_1')\delta(x_2')$ and

$$g(x_1') = \begin{cases} 1, & |x_1'| < \frac{1}{\sqrt{1-M^2}} \\ 0, & |x_1'| > \frac{1}{\sqrt{1-M^2}} \end{cases} \qquad \text{and} \qquad -\Delta' = -f.$$

Thus

$$u(z') = \frac{1}{2\pi}\int_{\mathbb{R}^2}\Big(-f(x')\Big)\log\Big(\frac{1}{|z'-x'|}\Big)\,dx' = -\frac{1}{2\pi}\int_{\mathbb{R}} g(x_1')\log\Big(\frac{1}{|(z_1', z_2') - (x_1', 0)|}\Big)\,dx_1',$$

and hence

$$u(z') = \frac{1}{2\pi}\int_{-\frac{1}{\sqrt{1-M^2}}}^{\frac{1}{\sqrt{1-M^2}}} \log\Big(|(z_1', z_2') - (x_1', 0)|\Big)\,dx_1'$$

so that

$$u(z_1', z_2') = u\Big(\frac{z_1}{\sqrt{1-M^2}}, z_2\Big) \qquad \text{gives} \qquad u = u(z_1, z_2).$$

5. See lecture notes

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