Mathematic Chalmers & GU

TMA372/MAN660: Partial Differential Equations, 2008–03–10

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Calculators, formula notes and other subject related material are not allowed.

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1. Prove the following error estimates for the linear interpolation error $v - \pi_1 v$ for the function v on the interval $J = (0, h)$:

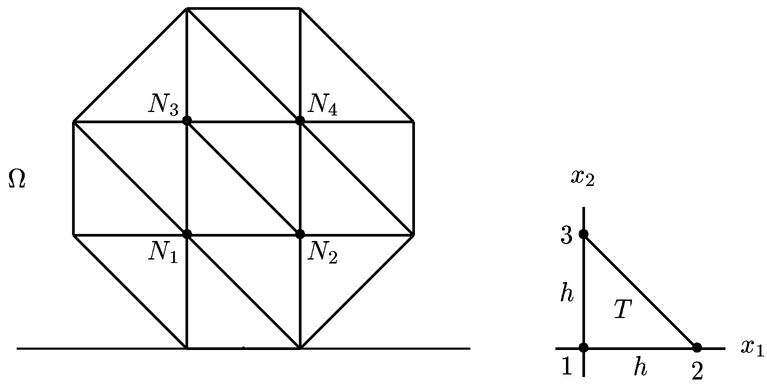
$$(a) \quad \max_{x\in J} |v(x) - \pi_1 v(x)| \le C_1 h^2 \max_{x\in J} |v''(x)|, \qquad (b) \quad \max_{x\in J} |v'(x) - (\pi_1 v)'(x)| \le C_2 h \max_{x\in J} |v''(x)|,$$

where $\pi_1 v(x) = ax + b$, $\pi_1 v(0) = v(0)$, $\pi_1 v(h) = v(h)$, $v' = dv/dx$. (c) Show that $C_1 \le 1/8$ and $C_2 \le 1/2$.

2. Formulate the cG(1) Galerkin finite element method for the boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega,$$

on the domain Ω. Write the matrices for the resulting equation system using the partition below (see fig.) with the nodes at N_1, N_2, N_3 and N_4 and a uniform mesh size h.
Hint: You may first compute the matrices for the reference triangle-element T.



3. Prove (a) an *a priori* and (b) an a posteriori error estimate for a finite element method for the boundary value problem, (the required interpolation estimates can be used without proofs):

$$-u_{xx} + u_x = f, \quad x \in (0,1); \qquad u(0) = u(1) = 0.$$

4. Consider the boundary value problem

$$u + a(x)u_x - \varepsilon u_{xx} = f, \quad x \in (0,1); \qquad u(0) = u_x(1) = 0,$$

where ε is a positive constant and $a(x)$ is a function of x such that $a \ge 0$ and $a_x(x) \ge 0$. Prove the following stability estimate for the solution u:

$$||\sqrt{\varepsilon} u_x|| + ||\sqrt{\varepsilon a_x} u_x|| + ||\varepsilon u_{xx}|| \le C||f||,$$

where $||\cdot||$ denotes the $L_2(I)$-norm, with $I = (0,1)$ and C is a constant.

5. Consider the following problem for the Klein-Gordon equation of quantum field theory:

$$\begin{cases} \ddot{u} - \Delta u + u = 0, & x \in \Omega \quad t > 0, \\ u = 0, & x \in \partial\Omega \quad t > 0, \\ u(x,0) = u_0(x), & \dot{u}(x,0) = u_1(x), \quad x \in \Omega. \end{cases}$$

(a) Define a suitable energy for this problem and show that the energy is conserved.

(b) Rewrite the equation as a system of two equations with time derivatives of order at most one, both in scalar and matrix form. Why is this reformulation needed?

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1. See the Book and Lecture notes.

2. Let V be the linear function space defined by

$$V := \{v : v \text{ is continuous in } \Omega,\ \ v = 0, \text{ on } \partial\Omega\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$-(\Delta u, \nabla v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v\, ds = (\nabla u, \nabla v), \qquad \forall v \in V.$$

Thus, since $v = 0$ on $\partial\Omega$, the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let now V_h be the usual finite element space consisting of continuous piecewise linear functions, on the given partition (triangulation), satisfying the boundary condition $v = 0$ on $\partial\Omega$:

$$V_h := \{v : v \text{ is continuous piecewise linear in } \Omega,\ \ v = 0, \text{ on } \partial\Omega\}.$$

The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^{4} \xi_i \varphi_j(x)$, where φ_j are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{4} \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\, dx + \int_\Omega \varphi_i \varphi_j\, dx \Big) = \int_\Omega f \varphi_i\, dx, \quad i = 1, 2, 3, 4$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1, x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1, x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\phi_2(x_1, x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\phi_3(x_1, x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2\, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2\, dx_1 dx_2 = \frac{h^2}{12},$$

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\, dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$\begin{aligned}
&M_{11} = M_{44} = 2m_{11} + 4m_{22} = \frac{1}{2}h^2, && S_{11} = S_{44} = 2s_{11} + 4s_{22} = 4,\\
&M_{22} = M_{33} = 3m_{11} + 2m_{22} = \frac{5}{12}h^2, && S_{22} = S_{33} = 3s_{11} + 2s_{22} = 4,\\
&M_{12} = M_{13} = M_{24} = M_{34} = 2m_{12} = \frac{1}{12}h^2, && S_{12} = S_{13} = S_{24} = S_{34} = 2s_{12} = -1,\\
&M_{23} = 2m_{23} = \frac{1}{12}h^2, && S_{23} = 2s_{23} = 0,\\
&M_{14} = 0, && S_{14} = 0,
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 6 & 1 & 1 & 0 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 0 & 1 & 1 & 6 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ 0 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

3. We multiply the differential equation by a test function $v \in H_0^1 = \{v : ||v|| + ||v'|| < \infty,\ v(0) = v(1) = 0\}$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

(1) $$\int_I (u'v' + u'v) = \int_I fv, \quad \forall v \in H_0^1(I).$$

Or equivalently, find $u \in H_0^1(I)$ such that

(2) $$(u_x, v_x) + (u_x, v) = (f, v), \quad \forall v \in H_0^1(I),$$

with $(\cdot,\cdot)$ denoting the $L_2(I)$ scalar product: $(u, v) = \int_I u(x)v(x)\,dx$. A *Finite Element Method* with $cG(1)$ reads as follows: Find $u_h \in V_h^0$ such that

(3) $$\int_I (u_h'v' + u_h'v) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I),$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Or equivalently, find $u_h \in V_h^0$ such that

(4) $$(u_{h,x}, v_x) + (u_{h,x}, v) = (f, v), \quad \forall v \in V_h^0.$$

Let now

$$a(u, v) = (u_x, v_x) + (u_x, v).$$

We want to show that $a(\cdot,\cdot)$ is both elliptic and continuous:

ellipticity

(5) $$a(u, u) = (u_x, u_x) + (u_x, u) = ||u_x||^2,$$

where we have used the boundary data, viz,

$$\int_0^1 u_x u \, dx = \Big[\frac{u^2}{2}\Big]_0^1 = 0.$$

continuity

$$a(u,v) = (u_x, v_x) + (u_x, v) \le ||u_x|| ||v_x|| + ||u_x|| ||v|| \le 2||u_x|| ||v_x||, \tag{6}$$

where we used the Poincare inequality $||v|| \le ||v_x||$.

Let now $e = u - u_h$, then (2)- (4) gives that

$$a(u - u_h, v) = (u_x - u_{h,x}, v_x) + (u_x - u_{h,x}, v) = 0, \quad \forall v \in V_h^0, \text{(Galerkin Orthogonality).} \tag{7}$$

A priori error estimate: We use ellipticity (5), Galerkin orthogonality (7), and the continuity (6) to get

$$\|u_x - u_{h,x}\|^2 = a(u - u_h, u - u_h) = a(u - u_h, u - v) \le 2\|u_x - u_{h,x}\| \|u_x - v_x\|, \quad \forall v \in V_h^0.$$

This gives that

$$\|u_x - u_{h,x}\| \le 2\|u_x - v_x\|, \quad \forall v \in V_h^0, \tag{8}$$

If we choose $v = \pi_h u \in V_h^0$, the interpolant of u, and use the interpolation estimate we get from (8) that

$$\|u_x - u_{h,x}\| \le 2\|u_x - (\pi u)_x\| \le 2C_i\|h u_{xx}\|. \tag{9}$$

A posteriori error estimate: We use again ellipticity (5), Galerkin orthogonality (7), and the variational formulation (1) to get

$$\begin{aligned} \|e_x\|^2 &= a(e,e) = a(e, e - \pi e) = a(u, e - \pi e) - a(u_h, e - \pi e) \\ &= (f, e - \pi e) - a(u_h, e - \pi e) = (f, e - \pi e) - (u_{h,x}, e_x - (\pi e)_x) - (u_{h,x}, e - \pi e) \\ &= (f - u_{h,x}, e - \pi e) \le C\|h(f - u_{h,x})\| \|e_x\|, \end{aligned} \tag{10}$$

where in the last equality we use the fact that $e(x_j) = (\pi e)(x_j)$, for j:s being the node points, also $u_{h,xx} \equiv 0$ on each $I_j := (x_{j-1}, x_j)$. Thus

$$(u_{h,x}, e_x - (\pi e)_x) = -\sum_j \int_{I_j} u_{h,xx}(e - \pi e) + \sum_j \Big(u_{h,x}(e - \pi e)\Big)\Big|_{I_j} = 0.$$

Hence, (10) yields:

$$\|e_x\| \le C\|h(f - u_{h,x})\|. \tag{11}$$

4. Multiply the equation by $-\varepsilon u_{xx}$ and integrate over $I = (0,1)$:

$$\int_0^1 -\varepsilon u u_{xx} + \int_0^1 -\varepsilon a(x) u_x u_{xx} + \int_0^1 \varepsilon^2 u_{xx}^2 = -\int_0^1 \varepsilon f u_{xx}. \tag{12}$$

We calculate the first two integral on the left hand side of (12)as:

$$\int_0^1 -\varepsilon u u_{xx} = -\Big[\varepsilon u u_x\Big]_0^1 + \int_0^1 \varepsilon u_x^2 = \int_0^1 \varepsilon u_x^2. \tag{13}$$

$$\int_0^1 -\varepsilon a(x) u_x u_{xx} = \Big[-\varepsilon a(x)\frac{u_x^2}{2}\Big] + \frac{1}{2}\int_0^1 \varepsilon a_x u_x^2 = \varepsilon a(0)\frac{u_x^2(0)}{2} + \frac{1}{2}\int_0^1 \varepsilon a_x u_x^2. \tag{14}$$

Inserting (13) and (14) in (13) yields

$$\begin{aligned} &\int_0^1 \varepsilon u_x^2 + \varepsilon a(0)\frac{u_x^2(0)}{2} + \frac{1}{2}\int_0^1 \varepsilon a_x u_x^2 + \int_0^1 \varepsilon^2 u_{xx}^2 \\ &\qquad = -\int_0^1 \varepsilon f u_{xx} \le \|f\| \|\varepsilon u_{xx}\| \le \|f\|^2 + \frac{1}{4}\|\varepsilon u_{xx}\|^2. \end{aligned} \tag{15}$$

Thus

$$\|\sqrt{\varepsilon} u_x\|^2 + \frac{1}{2}\|\sqrt{\varepsilon a_x} u_x\|^2 + \frac{3}{4}\|\varepsilon u_{xx}\|^2 \le \|f\|^2. \tag{16}$$

Hence

$$\|\sqrt{\varepsilon} u_x\| + \|\sqrt{\varepsilon a_x} u_x\| + \|\varepsilon u_{xx}\| \le C\|f\|. \tag{17}$$

5. a) Multiply the equation by $\dot{u}$ and integrate to obtain

$$(\ddot{u}, \dot{u}) - (\Delta u, \dot{u}) + (u, \dot{u}) = 0,$$
$$(\ddot{u}, \dot{u}) + (\nabla u, \nabla \dot{u}) + (u, \dot{u}) = 0,$$
$$\frac{1}{2}\frac{d}{dt}(||\dot{u}||^2 + ||\nabla u||^2 + ||u||^2) = 0,$$
$$\frac{1}{2}(||\dot{u}(t)||^2 + ||\nabla u(t)||^2 + ||u(t)||^2) = \frac{1}{2}(||u_1||^2 + ||\nabla u_0||^2 + ||u_0||^2).$$

This means that the energy $E = \frac{1}{2}(||\dot{u}(t)||^2 + ||\nabla u(t)||^2 + ||u(t)||^2)$ is conserved.
b) Set $v_1 = \dot{u}$, $v_2 = u$. Then

$$\dot{v}_1 - \Delta v_2 + v_2 = 0,$$
$$\dot{v}_2 - v_1 = 0.$$

Now we have a system $\dot{v} + Av = 0$ of first order in t and we can use various techniques developed for such systems, for example, we can apply standard time-discretization methods such as $dG(0)$ or $cG(1)$.

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