

TMA372/MAN660 Partial Differential Equations TM

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

1. Let $\pi_1 f$ be the linear interpolant of a twice continuously differentiable function f on the interval $I = (a, b)$. Prove that

$$\|f - \pi_1 f\|_{L_1(I)} \leq (b - a)^2 \|f''\|_{L_1(I)}.$$

2. (a) Derive the stiffness matrix and load vector in the global polynomial approximation $U(x) = \sum_{i=0}^q \xi_i t^i$ for the following ODE,

$$\dot{u}(t) = \lambda u(t), \quad 0 < t \leq 1, \quad u(0) = u_0.$$

- (b) Let $u_0 = 1$ and $\lambda = 2$ and determine the approximate solution $U(t)$, for $q = 1$ and $q = 2$.

3. Consider the Laplace equation with the Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & \text{in } \Omega = \{(x, y) : 0 < x < 1, 0 < y < 1\}, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

- a) Show that $\|D^2 u\| = \|\Delta u\|$, where $(D^2 u)^2 = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2$.

- b) Show the same result for the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ (instead of $u = 0$ on $\partial\Omega$).

- c) Show in the Dirichlet case (when $u = 0$ on $\partial\Omega$) that $\|u\| \leq C_\Omega \|\nabla u\|$ (Poincaré's inequality). What is the numerical value of the constant C_Ω ?

4. Let p be a positive constant. Prove an a priori and an a posteriori error estimate (in the H^1 -norm: $\|e\|_{H^1}^2 = \|e'\|^2 + \|e\|^2$) for a finite element method for problem

$$-u'' + pxu' + \left(1 + \frac{p}{2}\right)u = f, \quad \text{in } (0, 1), \quad u(0) = u(1) = 0.$$

5. Consider the initial boundary value problem for the heat equation

$$\begin{cases} \dot{u} - \Delta u = 0, & x \in \Omega \subset \mathbb{R}^2, & 0 < t \leq T, \\ u(x, t) = 0 & x \in \partial\Omega, & 0 < t \leq T, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

Prove the following stability estimates

$$\|u\|^2(T) + 2 \int_0^T \|\nabla u\|^2(t) dt = \|u_0\|^2, \quad (1)$$

$$\int_0^T t \|\Delta u\|^2(t) dt \leq \frac{1}{4} \|u_0\|^2, \quad (2)$$

$$\|\nabla u\|(T) \leq \frac{1}{\sqrt{2T}} \|u_0\|. \quad (3)$$

TMA372 Partial Differential Equations TM, 2005-12-13. Solutions

1. Let $\pi_1 f$ be the linear interpolant of a twice continuously differentiable function f on the interval $I = (a, b)$. Prove that

$$\|f - \pi_1 f\|_{L_1(I)} \leq (b - a)^2 \|f''\|_{L_1(I)}.$$

Solution: Let $\lambda_0(x) = \frac{\xi_1 - x}{\xi_1 - x_0}$ and $\lambda_1(x) = \frac{x - \xi_0}{\xi_1 - x_0}$ be two linear base functions. Then by the integral form of the Taylor formula we may write

$$\begin{cases} f(\xi_0) = f(x) + f'(x)(\xi_0 - x) + \int_x^{\xi_0} (\xi_0 - y) f''(y) dy, \\ f(\xi_1) = f(x) + f'(x)(\xi_1 - x) + \int_x^{\xi_1} (\xi_1 - y) f''(y) dy, \end{cases}$$

Therefore

$$\begin{aligned} \Pi_1 f(x) &= f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) \\ &= f(x) + \lambda_0(x) \int_x^{\xi_0} (\xi_0 - y) f''(y) dy + \lambda_1(x) \int_x^{\xi_1} (\xi_1 - y) f''(y) dy \end{aligned}$$

and by the triangle inequality we get

$$\begin{aligned} |f(x) - \Pi_1 f(x)| &= \left| \lambda_0(x) \int_x^{\xi_0} (\xi_0 - y) f''(y) dy + \lambda_1(x) \int_x^{\xi_1} (\xi_1 - y) f''(y) dy \right| \\ &\leq |\lambda_0(x)| \left| \int_x^{\xi_0} (\xi_0 - y) f''(y) dy \right| + |\lambda_1(x)| \left| \int_x^{\xi_1} (\xi_1 - y) f''(y) dy \right| \\ &\leq |\lambda_0(x)| \int_x^{\xi_0} |\xi_0 - y| |f''(y)| dy + |\lambda_1(x)| \int_x^{\xi_1} |\xi_1 - y| |f''(y)| dy \\ &\leq |\lambda_0(x)| \int_x^{\xi_0} (b - a) |f''(y)| dy + |\lambda_1(x)| \int_x^{\xi_1} (b - a) |f''(y)| dy \\ &\leq (b - a) (|\lambda_0(x)| + |\lambda_1(x)|) \int_a^b |f''(y)| dy \\ &= (b - a) (\lambda_0(x) + \lambda_1(x)) \int_a^b |f''(y)| dy = (b - a) \int_a^b |f''(y)| dy. \end{aligned}$$

Consequently

$$\int_a^b |f(x) - \Pi_1 f(x)| dx \leq \int_a^b (b - a) \left(\int_a^b |f''(y)| dy \right) dx = (b - a)^2 \|f''\|_{L_1(I)}.$$

2. (a) Derive the stiffness matrix and load vector in the global polynomial approximation $U(x) = \sum_{i=0}^q \xi_i t^i$ for the following ODE,

$$\dot{u}(t) = \lambda u(t), \quad 0 < t \leq 1, \quad u(0) = u_0.$$

(b) Let $u_0 = 1$ and $\lambda = 2$ and determine the approximate solution $U(t)$, for $q = 1$ and $q = 2$.

Solution: (a) We insert U in the equation (note $\dot{U}(t) = \sum_{j=1}^q j \xi_j t^{j-1}$), multiplying the result by t^i , $i = 1, \dots, q$ and integrating over $(0, 1)$ we get

$$(1) \quad \sum_{j=1}^q \xi_j \int_0^1 \left(j t^{i+j-1} - \lambda t^{i+j} \right) dt = \lambda \xi_0 \int_0^1 t^i dt,$$

where the contribution for $j = 0$ which corresponds to $\xi_0 = u(0)$ is on the right hand side. After integration we get

$$(2) \quad \sum_{j=1}^q \left(\frac{j}{j+i} - \frac{\lambda}{j+i+1} \right) \xi_j = \frac{\lambda}{i+1} \xi_0, \quad i, j = 1, \dots, q. \iff A\xi = b, \quad \text{where}$$

$$\begin{cases} a_{ij} = \frac{j}{j+i} - \frac{\lambda}{j+i+1}, & i, j = 1, \dots, q. \\ b_i = \frac{\lambda}{i+1} \xi_0, & i = 1, \dots, q. \end{cases}$$

(b) Thus with $u_0 = 1$ and $\lambda = 2$ we get

$$A\xi = b, \quad \text{where} \quad \begin{cases} a_{ij} = \frac{j}{j+i} - \frac{2}{j+i+1}, & i, j = 1, \dots, q. \\ b_i = \frac{2}{i+1}, & i = 1, \dots, q. \end{cases}$$

$$q = 1: \implies -\frac{1}{6}\xi_1 = 1 \implies \xi_1 = -6. \implies U(t) = 1 - 6t.$$

$$q = 2: \implies \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \iff \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Hence

$$U(t) = 1 + t + 5t^2.$$

3. Consider the Laplace equation with the Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & \text{in } \Omega = \{(x, y) : 0 < x < 1, 0 < y < 1\}, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

a) Show that $\|D^2u\| = \|\Delta u\|$, where $(D^2u)^2 = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2$.

b) Show the same result for the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ (instead of $u = 0$ on $\partial\Omega$).

c) Show in the Dirichlet case (when $u = 0$ on $\partial\Omega$) that $\|u\| \leq C_\Omega \|\nabla u\|$ (Poincaré's inequality). What is the numerical value of the constant C_Ω ?

Solution: (a) *A general approach* Let $\Gamma = \partial\Omega$ be the boundary of Ω . We have that

$$\|\Delta u\|^2 = \int_{\Omega} u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}.$$

Now an application of the Green's formula (partial integration first in y and then in x) gives

$$\int_{\Omega} u_{xx}u_{yy} = \int_{\Gamma} u_{xx}u_y n_y - \int_{\Omega} u_{xxy}u_y = \int_{\Gamma} u_{xx}u_y n_y - \int_{\Gamma} u_{xy}u_y n_x + \int_{\Omega} u_{xy}u_{xy},$$

where $n = (n_x, n_y)$ is the outward unit normal at the boundary. Now, on the part of the boundary Γ , where $n_y \neq 0$, we have $u_{xx} = 0$, since $u = 0$. Likewise, $u_y = 0$ on the part of the boundary, where $n_x \neq 0$. Thus $\int_{\Omega} u_{xx}u_{yy} = \int_{\Omega} u_{xy}u_{xy}$, which gives the desired identity.

Alternatively, let $\Omega = [0, 1] \times [0, 1]$. Then

$$\|\Delta u\| = \int_{\Omega} (\Delta u)^2 = \int_{\Omega} (u_{xx} + u_{yy})^2 = \int_{\Omega} (u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}).$$

$$\int_{\Omega} u_{xx}u_{yy} = \int_{\Gamma} u_x u_{yy} n_1 - \int_{\Omega} u_{xy}u_{yx} = \int_{\Gamma} u_x u_{yy} n_1 - \int_{\Gamma} u_x u_{xy} n_2 + \int_{\Omega} u_{xy}u_{xy},$$

where we have used partial integration and the identities $u_{yyx} = u_{yxy} = u_{xyy}$. Further we have that

$$-\int_{\Gamma} u_x u_{xy} n_2 = -\int_{\Gamma} u_x (u_y n_2)_x = -\int_{\Gamma} u_x (-u|n_2|)_x = \int_{\Gamma} u_x^2 |n_2| \geq 0,$$

where we have used the boundary condition $n \cdot \nabla u = -u$. Note that $n_2 = 0$ on the vertical boundary lines. Similarly we get

$$\int_{\Gamma} u_x u_{xy} n_1 = \int_{\Gamma} u_{yy} (-u) |n_1|,$$

which after integration by parts over the vertical boundary lines (where $n_1 \neq 0$) gives that

$$\begin{aligned} & \int_{\Gamma} u_y u_y |n_1| - u_y(1,1)u(1,1) + u_y(1,0)u(1,0) - u_y(0,1)u(0,1) + u_y(0,0)u(0,0) \\ &= \int_{\Gamma} u_y u_y |n_1| + u(1,1)u(1,1) + u(1,0)u(1,0) + (0,1)u(0,1) + u(0,0)u(0,0) \geq 0, \end{aligned}$$

where we have again used the boundary condition $n \cdot \nabla u = -u$. In other words we have proved that

$$\|\Delta u\| \geq \|D^2 u\|.$$

The reversed inequality is trivial.

(b) In the case of Neumann boundary condition: $\frac{\partial u}{\partial n} = 0$ on the boundary, we have that $u_y = 0$ on the part of Γ where $n_y \neq 0$, similarly $u_{xy} = 0$ on the part of Γ where $n_x \neq 0$ (because, then $u_x = 0$ in y -direction). Thus we obtain the same identity as in (a).

(c) See proof of Poincaré inequality in Chapter 15.

4. Let p be a positive constant. Prove an a priori and an a posteriori error estimate (in the H^1 -norm: $\|e\|_{H^1}^2 = \|e'\|^2 + \|e\|^2$) for a finite element method for problem

$$-u'' + pxu' + \left(1 + \frac{p}{2}\right)u = f, \quad \text{in } (0, 1), \quad u(0) = u(1) = 0.$$

Solution: We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0, 1)$ and integrate over I . Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$(3) \quad \int_I \left(u'v' + pxu'v + \left(1 + \frac{p}{2}\right)uv \right) = \int_I fv, \quad \forall v \in H_0^1(I).$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$(4) \quad \int_I \left(U'v' + pxU'v + \left(1 + \frac{p}{2}\right)Uv \right) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I),$$

where

$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I, v(0) = v(1) = 0\}$.

Now let $e = u - U$, then (1)-(2) gives that

$$(5) \quad \int_I \left(e'v' + pxe'v + \left(1 + \frac{p}{2}\right)ev \right) = 0, \quad \forall v \in V_h^0.$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$(6) \quad \int_I pxe'e = \frac{p}{2} \int_I x \frac{d}{dx}(e^2) = \frac{p}{2}(xe^2)|_0^1 - \frac{p}{2} \int_I e^2 = -\frac{p}{2} \int_I e^2,$$

so that

$$(7) \quad \begin{aligned} \|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I \left(e'e' + pxe'e + \left(1 + \frac{p}{2}\right)ee \right) \\ &= \int_I \left((u-U)'e' + px(u-U)'e + \left(1 + \frac{p}{2}\right)(u-U)e \right) = \{v = e \text{ in (1)}\} \\ &= \int_I fe - \int_I \left(U'e' + pxU'e + \left(1 + \frac{p}{2}\right)Ue \right) = \{v = \pi_h e \text{ in (2)}\} \\ &= \int_I f(e - \pi_h e) - \int_I \left(U'(e - \pi_h e)' + pxU'(e - \pi_h e) + \left(1 + \frac{p}{2}\right)U(e - \pi_h e) \right) \\ &= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e), \end{aligned}$$

where $\mathcal{R}(U) := f + U'' - pxU' - \left(1 + \frac{p}{2}\right)U = f - pxU' - \left(1 + \frac{p}{2}\right)U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$\begin{aligned} \|e\|_{H^1}^2 &\leq \|h\mathcal{R}(U)\| \|h^{-1}(e - \pi_h e)\| \\ &\leq C_i \|h\mathcal{R}(U)\| \|e'\| \leq C_i \|h\mathcal{R}(U)\| \|e\|_{H^1}, \end{aligned}$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \leq C_i \|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (4) and write

$$\begin{aligned} \|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I \left(e'e' + pxe'e + \left(1 + \frac{p}{2}\right)ee \right) \\ &= \int_I \left(e'(u-U)' + pxe'(u-U) + \left(1 + \frac{p}{2}\right)e(u-U) \right) = \{v = U - \pi_h u \text{ in (3)}\} \\ &= \int_I \left(e'(u - \pi_h u)' + pxe'(u - \pi_h u) + \left(1 + \frac{p}{2}\right)e(u - \pi_h u) \right) \\ &\leq \|(u - \pi_h u)'\| \|e'\| + p\|u - \pi_h u\| \|e'\| + \left(1 + \frac{p}{2}\right)\|u - \pi_h u\| \|e\| \\ &\leq \{ \|(u - \pi_h u)'\| + (1+p)\|u - \pi_h u\| \} \|e\|_{H^1} \\ &\leq C_i \{ \|hu''\| + (1+p)\|h^2u''\| \} \|e\|_{H^1}, \end{aligned}$$

this gives that

$$\|e\|_{H^1} \leq C_i \{ \|hu''\| + (1+p)\|h^2u''\| \},$$

which is the a priori error estimate.

5. Consider the initial boundary value problem for the heat equation

$$\begin{cases} \dot{u} - \Delta u = 0, & x \in \Omega \subset \mathbb{R}^2, & 0 < t \leq T, \\ u(x, t) = 0 & x \in \partial\Omega, & 0 < t \leq T, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

Prove the following stability estimates

$$(8) \quad \|u\|^2(T) + 2 \int_0^T \|\nabla u\|^2(t) dt = \|u_0\|^2,$$

$$(9) \quad \int_0^T t \|\Delta u\|^2(t) dt \leq \frac{1}{4} \|u_0\|^2,$$

$$(10) \quad \|\nabla u\|(T) \leq \frac{1}{\sqrt{2T}} \|u_0\|.$$

Solution See Lecture Notes or text book Chapter 16.

MA