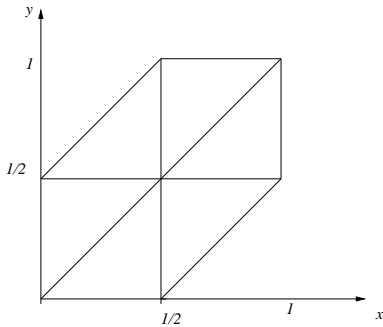


TMA372/MAN660 Partiella differentialekvationer TM, IMP, E3, GU

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

1. Let Ω be the triangulated domain below. Compute the cG(1) solution of

$$\begin{cases} -\Delta u = 1, & \text{on } \Omega \\ u_x(1, y) = 0, \quad 1/2 \leq y \leq 1, & u(x, y) = 0, \quad \text{on the rest of boundary.} \end{cases}$$



2. Consider the initial value problem $\dot{u}(t) + a(t)u(t) = f(t)$, for $0 < t < T$, and $u(0) = u_0$. Prove the stability estimates

$$|u(t)| \leq e^{-\alpha t} |u_0| + \frac{1}{\alpha} (1 - e^{-\alpha t}) \max_{0 \leq s \leq t} |f(s)|, \quad a(t) \geq \alpha > 0, \quad \text{and}$$

$$|u(t)| \leq |u_0| + \int_0^t |f(s)| ds, \quad a(t) \geq 0.$$

3. Prove that if $u = 0$ on the boundary of the unit square Ω , then

$$\left(\int_{\Omega} |u|^2 dx \right)^{1/2} \leq \left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}.$$

4. Prove an a priori and an a posteriori error estimate (in the energy norm: $\|u\|_E^2 := \|u'\|^2 + \|u\|^2$) for the cG(1) finite element method for the problem

$$\begin{cases} -u'' + \alpha u' + u = f, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

where $\alpha \geq 0$. For which value of α is the a priori error estimate optimal?

5. Consider the boundary value problem

$$\begin{cases} -\Delta u = 0, & \text{in a bounded domain } \Omega \subset \mathbb{R}^d, \quad d = 2, 3. \\ \frac{\partial u}{\partial n} + u = g, & \text{on } \Gamma = \partial\Omega. \end{cases}$$

- a) Prove the L_2 stability estimate

$$\|\nabla u\|_{L_2(\Omega)} + \frac{1}{2} \|u\|_{L_2(\Gamma)} \leq \frac{1}{2} \|g\|_{L_2(\Gamma)}.$$

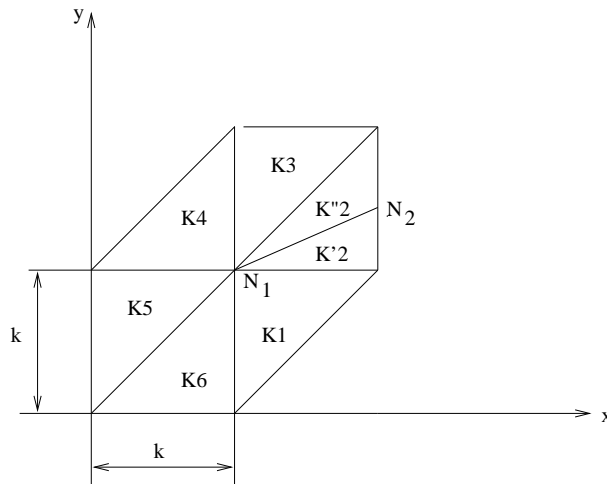
- b) Verify the conditions on Riesz/Lax-Milgram theorems for this problem.

**TMA371/MAN660 Partial Differential Equations TM, IMP, E3, GU
2002-12-17. Solutions**

1.

Let Ω be the triangulated domain below. Compute the cG(1) solution of

$$\begin{cases} -\Delta u = 1, & \text{on } \Omega \\ u_x(1, y) = 0, \quad 1/2 \leq y \leq 1, & u(x, y) = 0, \quad \text{on the rest of boundary.} \end{cases}$$



Solution. We split the boundary $\Gamma := \partial\Omega$ of the domain as $\Gamma = \Gamma_N + \Gamma_D$ with $\Gamma_N = \{(1, y) : 1/2 \leq y \leq 1\}$ and $\Gamma_D = \Gamma \setminus \Gamma_N$.

Variational Formulation: Using Green's formula we have that

$$\begin{aligned} \int_{\Omega} 1 \cdot v &= \int_{\Omega} -\Delta u v = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Gamma} (\partial_n u) v \\ (1) \quad &= \{v = 0 \text{ on } \Gamma_D, \text{ and } \partial_n u = u_x = 0 \text{ on } \Gamma_N\} \\ &= \int_{\Omega} \nabla u \cdot \nabla v. \end{aligned}$$

Thus we have the finite element formulation: Find piecewise linear function $U \in V_h$ such that

$$(2) \quad \int_{\Omega} \nabla U \cdot \nabla v = \int_{\Omega} 1 \cdot v, \quad \forall v \in V_h.$$

Remark. It is natural to think $P_1 = (1/2, 1/2)$, $P_2 = (1, 1/2)$ and $P_3 = (1, 1)$ as nodes. Then there are two possibilities:

(I) Normally we let the basis functions ψ_2 , and ψ_3 (corresponding to the nodes P_2 and P_3 , respectively) to have $K_1 \cap K_2$, and $K_2 \cap K_3$ as their supports, respectively. This however extends the Neumann boundary condition from Γ_N to $\Gamma_N \cap \{(x, y) : y = x - 1/2, 1/2 \leq x \leq 1\} \cup \{(x, y) : y = 1, 1/2 \leq x \leq 1\}$, thus removing the Dirichlet condition from the line-segments $\{(x, y) : y = x - 1/2, 1/2 \leq x \leq 1\}$ and $\{(x, y) : y = 1, 1/2 \leq x \leq 1\}$.

(II) To circumvent this difficulty, we restrict the support of both basis functions ψ_i , $i = 2, 3$ to ONLY! K_2 . And this choice introduce discontinuities and cannot be considered in cG(1).

Hence summing up, because of cG(1) approximation in this problem we cannot choose the boundary points $(1, 1/2)$ and $(1, 1)$ as nodes (this will either destroy the continuity or replace the Dirichlet condition on $\{(x, y) : 1/2 \leq x \leq 1\}$, with $0 \leq y \leq 1/2$ and $y = 1$ by the Neumann condition. So the only adequate way is the *refinement* of K_2 into two triangles K'_2 and K''_2 , by letting, e.g. $N_2 = (1, 3/4)$, as in the figure. We define the test functions $\varphi_i(N_j) = \delta_{ij}$, $i, j = 1, 2$. In order to have zero boundary condition on Γ_D , φ_2 , has $K_2 = K'_2 \cap K''_2$ as its support. Let now

$$U(x, y) = U_1\varphi_1(x, y) + U_2\varphi_2(x, y).$$

where $U_i = U(N_i)$, $i = 1, 2$. Observe that φ_i s are the bases functions for V_h and thus the equation (2) is equivalent to the following system:

$$(3) \quad \int_{\Omega} \nabla\varphi_1 \cdot \nabla\varphi_i U_1 + \int_{\Omega} \nabla\varphi_2 \cdot \nabla\varphi_i U_2 + = \int_{\Omega} \varphi_i, \quad i = 1, 2.$$

Let now $\varphi'_2 = \varphi|_{K'_2}$ and $\varphi''_2 = \varphi|_{K''_2}$, then using a standard triangle K_2 with vertices $(0, 0)$, $(k, 0)$, and (k, k) , for $\varphi'_2(x, y) = ax + by + c$ we have that

$$\begin{aligned} \varphi'_2(0, 0) &= 0 \Rightarrow c = 0, \\ \varphi'_2(k, 0) &= 0 \Rightarrow ak = 0 \Leftrightarrow a = 0, \\ \varphi'_2(k, k/2) &= 1 \Rightarrow bk/2 = 1 \Leftrightarrow b = 2/k. \end{aligned}$$

Thus $\varphi'_2(x, y) = \frac{2}{k}y$. Similarly for $\varphi''_2(x, y) = ax + by + c$ we have that

$$\begin{aligned} \varphi''_2(0, 0) &= 0 \Rightarrow c = 0, \\ \varphi''_2(k, k) &= 0 \Rightarrow ak + bk = 0 \Leftrightarrow b = -a, \\ \varphi''_2(k, k/2) &= 1 \Rightarrow ak + bk/2 = 1 \Leftrightarrow a = 2/k, b = -2/k. \end{aligned}$$

Thus $\varphi''_2(x, y) = \frac{2}{k}x - \frac{2}{k}y$.

Considering in standard triangles we can easily see from the figure that:

$$\begin{aligned} \nabla\varphi_1 \Big|_{K_1} &= \frac{1}{k}(1, 1), & \nabla\varphi_2 \Big|_{K_1} &= (0, 0), \\ \nabla\varphi_1 \Big|_{K_2} &= \frac{1}{k}(-1, 0), & \nabla\varphi_2 \Big|_{K'_2} &= \frac{1}{k}(0, 2), & \nabla\varphi_2 \Big|_{K''_2} &= \frac{1}{k}(2, -2), \\ \nabla\varphi_1 \Big|_{K_3} &= \frac{1}{k}(0, -1), & \nabla\varphi_2 \Big|_{K_3} &= (0, 0), \\ \nabla\varphi_1 \Big|_{K_4} &= \frac{1}{k}(1, -1), & \nabla\varphi_2 \Big|_{K_4} &= (0, 0), \\ \nabla\varphi_1 \Big|_{K_5} &= \frac{1}{k}(1, 0), & \nabla\varphi_2 \Big|_{K_5} &= (0, 0), \\ \nabla\varphi_1 \Big|_{K_6} &= \frac{1}{k}(0, 1), & \nabla\varphi_2 \Big|_{K_6} &= (0, 0). \end{aligned}$$

Now considering the intersections of the supports of the φ functions we have that:

$$\begin{aligned} \int_{\Omega} \nabla \varphi_1 \cdot \nabla \varphi_1 &= \sum_{j=1}^6 \int_{K_j} \nabla \varphi_1 \cdot \nabla \varphi_1 \\ &= \frac{k^2}{2} \frac{1}{k^2} \left\{ (-1, 1) \cdot (-1, 1) + (-1, 0) \cdot (-1, 0) + (0, -1) \cdot (0, -1) \right. \\ &\quad \left. + (1, -1) \cdot (1, -1) + (1, 0) \cdot (1, 0) + (1, 0) \cdot (1, 0) \right\} \\ &= \frac{1}{2} \{2 + 1 + 1 + 2 + 1 + 1\} = 4. \end{aligned}$$

Note that $|K'| = |K''| = \frac{1}{2}k \cdot \frac{k}{2} = \frac{k^2}{4}$, thus

$$\begin{aligned} \int_{\Omega} \nabla \varphi_2 \cdot \nabla \varphi_2 &= \int_{K_2} \nabla \varphi_2 \cdot \nabla \varphi_2 = \int_{K'_2} \nabla \varphi'_2 \cdot \nabla \varphi'_2 + \int_{K''_2} \nabla \varphi''_2 \cdot \nabla \varphi''_2 \\ &= \frac{k^2}{4} \frac{1}{k^2} \left\{ (0, 2) \cdot (0, 2) + (2, -2) \cdot (2, -2) \right\} = \frac{1}{4} \{4 + 8\} = 3, \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} \nabla \varphi_1 \cdot \nabla \varphi_2 &= \int_{K_2} \nabla \varphi_1 \cdot \nabla \varphi_2 = \int_{K'_2} \nabla \varphi_1 \cdot \nabla \varphi'_2 + \int_{K''_2} \nabla \varphi_1 \cdot \nabla \varphi''_2 \\ &= \frac{k^2}{4} \frac{1}{k^2} \left\{ (-1, 0) \cdot (0, 2) + (-1, 0) \cdot (2, -2) \right\} = \frac{1}{4} \{-2\} = -1/2. \end{aligned}$$

As for the right hand side in (3) we have

$$\int_{\Omega} \varphi_1 = 6 \cdot \frac{1}{3} \frac{k^2}{2} = k^2, \quad \int_{\Omega} \varphi_2 = \int_{K_2} \varphi_2 = \dots = 1 \cdot \frac{1}{3} \frac{k^2}{2} = k^2/6.$$

Thus recalling that $k = 1/2$, we have the following system of equations:

$$\begin{pmatrix} 4 & -1/2 \\ -1/2 & 3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} 1/4 \\ 1/24 \end{pmatrix} \Rightarrow U_1 = 6U_2 - 1/12, \text{ with } U_2 = \frac{7}{6(47)}.$$

2. See stability lemma in chapter 9 of the lecture notes.

3. Prove that if $u = 0$ on the boundary of the unit square Ω , then

$$\left(\int_{\Omega} |u|^2 dx \right)^{1/2} \leq \left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}.$$

Solution. We have that

$$\begin{aligned} |u(x)| &= |u(x_1, x_2) - u(0, x_2)| = \left| \int_0^{x_1} \frac{\partial}{\partial x_1} u(\bar{x}_1, x_2) d\bar{x}_1 \right| \\ &= \left| \int_0^{x_1} 1 \cdot \frac{\partial}{\partial x_1} u(\bar{x}_1, x_2) d\bar{x}_1 \right| \leq \{ \text{Cauchy's inequality} \} \\ &\leq \left(\int_0^{x_1} 1^2 d\bar{x}_1 \right)^{1/2} \cdot \left(\int_0^{x_1} \left(\frac{\partial}{\partial x_1} u(\bar{x}_1, x_2) \right)^2 d\bar{x}_1 \right)^{1/2} \\ &\leq \left(\int_0^1 \left(\frac{\partial}{\partial x_1} u(\bar{x}_1, x_2) \right)^2 d\bar{x}_1 \right)^{1/2}. \end{aligned}$$

This implies that

$$\begin{aligned}
\int_{\Omega} |u|^2 dx &\leq \int_{\Omega} \left(\int_0^1 \left(\frac{\partial}{\partial x_1} u(\bar{x}_1, x_2) \right)^2 d\bar{x}_1 \right) dx \\
&= \int_0^1 \int_0^1 \left(\int_0^1 \left(\frac{\partial}{\partial x_1} u(\bar{x}_1, x_2) \right)^2 d\bar{x}_1 \right) dx_1 dx_2 \\
&= \int_0^1 \left(\int_0^1 \left(\frac{\partial}{\partial x_1} u(\bar{x}_1, x_2) \right)^2 d\bar{x}_1 \right) dx_2 = \int_0^1 \int_0^1 \left(\frac{\partial}{\partial x_1} u(x_1, x_2) \right)^2 dx_1 dx_2 \\
&= \int_{\Omega} \left(\frac{\partial}{\partial x_1} u(x_1, x_2) \right)^2 dx \leq \int_{\Omega} |\nabla u|^2 dx,
\end{aligned}$$

which gives the desired result:

$$\left(\int_{\Omega} |u|^2 dx \right)^{1/2} \leq \left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}.$$

4. Prove an a priori and an a posteriori error estimate (in the energy norm: $\|u\|_E^2 := \|u'\|^2 + \|u\|^2$) for the cG(1) finite element method for the problem

$$\begin{cases} -u'' + \alpha u' + u = f, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

where $\alpha \geq 0$. For which value of α is the a priori error estimate optimal?

Solution. The Variational formulation:

Multiply the equation by $v \in V$, integrate by parts over $(0, 1)$ and use the boundary conditions to obtain

$$(4) \quad \text{Find } u \in V : \int_0^1 u'v' dx + \int_0^1 \alpha u'v dx + \int_0^1 uv dx = \int_0^1 fv dx, \quad \forall v \in V.$$

cG(1):

$$(5) \quad \text{Find } U \in V_h : \int_0^1 U'v' dx + \int_0^1 \alpha U'v dx + \int_0^1 Uv dx = \int_0^1 fv dx, \quad \forall v \in V_h.$$

From (1)-(2), we find The Galerkin orthogonality:

$$(6) \quad \int_0^1 \left((u - U)'v' + \alpha(u - U)'v + (u - U)v \right) dx = 0, \quad \forall v \in V_h.$$

We define the inner product $(\cdot, \cdot)_E$ associated to the energy norm to be

$$(v, w)_E = \int_0^1 (v'w' + vw) dx, \quad \forall v, w \in V.$$

Note that

$$(7) \quad \int_0^1 e'e dx = \frac{1}{2} \int_0^1 \frac{d}{dx} (e^2) dx = \frac{1}{2} [e^2]_0^1 = 0.$$

Thus using (7) we have

$$(8) \quad \|e\|_E^2 = \int_0^1 (e'e' + ee) dx = \int_0^1 (e'e' + \alpha e'e + ee) dx.$$

We split the second factor e as $e = u - U = u - v + v - U$, with $v \in V_h$ and write

$$\begin{aligned}
\|e\|_E^2 &= \int_0^1 \left(e'(u - U)' + \alpha e'(u - U) + e(u - U) \right) dx = \left\{ v \in V_h \right\} \\
&= \int_0^1 \left(e'(u - v)' + \alpha e'(u - v) + e(u - v) \right) dx \\
(9) \quad &+ \int_0^1 \left(e'(v - U)' + \alpha e'(v - U) + e(v - U) \right) dx \\
&= \int_0^1 \left(e'(u - v)' + \alpha e'(u - v) + e(u - v) \right) dx,
\end{aligned}$$

where, in the last step, we have used the Galerkin orthogonality to eliminate terms involving U . Now we can write

$$\begin{aligned}
\|e\|_E^2 &= \int_0^1 \left(e'(u - v)' + e(u - v) + \alpha e'(u - v) \right) dx \\
(10) \quad &\leq \|e\|_E \cdot \|u - v\|_E + \alpha \|e'\|_{L_2} \|u - v\|_{L_2} \\
&\leq \|e\|_E \left(\|u - v\|_E + \alpha \|u - v\|_{L_2} \right) \leq \|e\|_E \|u - v\|_E (1 + \alpha),
\end{aligned}$$

and derive the a priori error estimate:

$$\|e\|_E \leq \|u - v\|_E (1 + \alpha), \quad \forall v \in V_h.$$

To obtain a posteriori error estimates the idea is to eliminate u -terms, by using the differential equation, and replacing their contributions by the data f . Then this f combined with the remaining U -terms would yield to the residual error:

A posteriori error estimate:

$$\begin{aligned}
\|e\|_E^2 &= \int_0^1 (e'e' + ee) dx = \int_0^1 (e'e' + \alpha e'e + ee) dx \\
(11) \quad &= \int_0^1 (u'e' + \alpha u'e + ue) dx - \int_0^1 (U'e' + \alpha U'e + Ue) dx.
\end{aligned}$$

Now using the variational formulation (4) we have that

$$\int_0^1 (u'e' + \alpha u'e + ue) dx = \int_0^1 fe dx.$$

Inserting in (11) and using (5) with $v = \Pi_h e$ we get

$$\begin{aligned}
\|e\|_E^2 &= \int_0^1 fe dx - \int_0^1 (U'e' + \alpha U'e + Ue) dx \\
(12) \quad &+ \int_0^1 (U'\Pi_h e' + \alpha U'\Pi_h e + U\Pi_h e) dx - \int_0^1 f\Pi_h e dx.
\end{aligned}$$

Thus

$$\begin{aligned}
\|e\|_E^2 &= \int_0^1 f(e - \Pi_h e) \, dx - \int_0^1 \left(U'(e - \Pi_h e)' + \alpha U'(e - \Pi_h e) + U(e - \Pi_h e) \right) dx \\
&= \int_0^1 f(e - \Pi_h e) \, dx - \int_0^1 (\alpha U' + U)(e - \Pi_h e) \, dx - \sum_{j=1}^{M+1} \int_{I_j} U'(e - \Pi_h e)' \, dx \\
&= \{\text{partial integration}\} \\
&= \int_0^1 f(e - \Pi_h e) \, dx - \int_0^1 (\alpha U' + U)(e - \Pi_h e) \, dx + \sum_{j=1}^{M+1} \int_{I_j} U''(e - \Pi_h e) \, dx \\
&= \int_0^1 (f + U'' - \alpha U' - U)(e - \Pi_h e) \, dx = \int_0^1 R(U)(e - \Pi_h e) \, dx \\
&= \int_0^1 hR(U)h^{-1}(e - \Pi_h e) \, dx \leq \|hR(U)\|_{L_2} \|h^{-1}(e - \Pi_h e)\|_{L_2} \\
&\leq C_i \|hR(U)\|_{L_2} \cdot \|e'\|_{L_2} \leq \|hR(U)\|_{L_2} \cdot \|e\|_E.
\end{aligned}$$

This gives the a posteriori error estimate:

$$\|e\|_E \leq C_i \|hR(U)\|_{L_2},$$

with $R(U) = f + U'' - \alpha U' - U = f - \alpha U' - U$ on (x_{i-1}, x_i) , $i = 1, \dots, M+1$.

The a priori error estimate is optimal for $\alpha = 0$.

5. Consider the boundary value problem

$$\begin{cases} -\Delta u = 0, & \text{in a bounded domain } \Omega \subset \mathbb{R}^d, \, d = 2, 3. \\ \frac{\partial u}{\partial n} + u = g, & \text{on } \Gamma = \partial\Omega. \end{cases}$$

a) Prove the L_2 stability estimate

$$\|\nabla u\|_{L_2(\Omega)}^2 + \frac{1}{2}\|u\|_{L_2(\Gamma)}^2 \leq \frac{1}{2}\|g\|_{L_2(\Gamma)}^2.$$

b) Verify the conditions on Riesz/Lax-Milgram theorems for this problem.

Solution: a) Using Greens formula we have that

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega} \nabla u \cdot \nabla u = - \int_{\Omega} (\Delta u)u + \int_{\partial\Omega} \frac{\partial u}{\partial n} u = \int_{\partial\Omega} (g - u)u.$$

In other words

$$\|\nabla u\|_{L_2(\Omega)}^2 + \|u\|_{L_2(\Gamma)}^2 = \int_{\partial\Omega} gu \leq \|g\|_{L_2(\Gamma)}^2 \|u\|_{L_2(\Gamma)}^2 \leq \frac{1}{2}\|g\|_{L_2(\Gamma)}^2 + \frac{1}{2}\|u\|_{L_2(\Gamma)}^2,$$

which gives the desired estimate.

To show the Riesz/Lax-Milgram conditions we introduce the notation

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial\Omega} uv, \quad \text{and} \quad L(v) = \int_{\partial\Omega} gv.$$

Then $a(u, v)$ is a scalar product with the corresponding norm $\|v\|_a = a(v, v)^{1/2}$.

For instance we have that $\|v\|_a = 0$, only if $v = 0$:

$$0 = \|v\|_a^2 = a(u, v) = \int_{\Omega} |\nabla v|^2 + \int_{\partial\Omega} v^2 \geq \alpha \int_{\Omega} v^2, \quad \text{for some } \alpha > 0 \Rightarrow v = 0.$$

Further $L(v)$ is bounded in this norm, e.g. if $\|g\|_{\partial\Omega} < \infty$, then

$$|L(v)| \leq \|g\|_{\partial\Omega} \|v\|_{\partial\Omega} \leq \|g\|_{\partial\Omega} \|v\|_a.$$

We can also apply Riesz theorem in the sense that there exists u such that

$$a(u, v) = L(v), \quad \forall v,$$

and u is uniquely determined by

$$\|u\|_a = \|g\|_{\partial\Omega}.$$

Moreover since

$$a(u, v) = - \int_{\Omega} \Delta u v + \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} + u \right) v,$$

we have that

$$\Delta u = 0, \quad \text{in } \Omega \quad \frac{\partial u}{\partial n} + u = g \quad \text{on } \Gamma.$$

MA