Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2014–08–27, 8:30-12:30

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: 3: 15-20p, 4: 21-27p och 5: 28p- For GU studentsG:15-24p, VG: 25p-
For solutions and information about gradings see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1314/index.html.

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1. Show the following estimate for the linear interpolation $\pi_1 f$ of a function $f \in C^2(0,1)$,

$$||\pi_1 f - f||_{L_\infty(0,1)} \le \frac{1}{8} \max_{0\le\xi\le 1} |f''(\xi)|.$$

2. Prove an a priori and an a posteriori error estimate for the cG(1) finite element method for

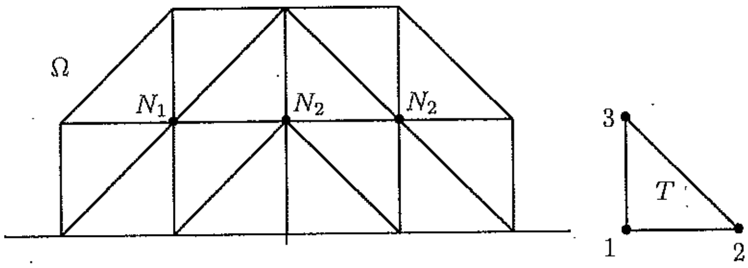
$$\begin{cases} -u''(x) + xu'(x) + u(x) = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

in the energy norm $||v||_E$ with $||v||_E^2 = ||v||^2_{L_2(I)} + ||v'||^2_{L_2(I)}$.

3. Formulate the cG(1) piecewise continuous Galerkin method for the boundary value problem

$$\begin{cases} -\Delta u + u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

on the domain Ω (see fig.) Write the matrices for the resulting equation system using the following mesh with nodes at N_1, N_2 and N_3 and a uniform mesh size h.
Hint: You may first compute the matrices for a standard triangle-element T.



4. Prove that if $u = 0$ on the boundary of the unit square $\Omega = [0,1] \times [0,1]$, then

$$||u|| \le ||\nabla u||, \qquad \text{(A Poincare inequality in 2D)}, \qquad ||w|| = \Big(\int_\Omega |w|^2\, dx\Big)^{1/2}.$$

5. Consider the boundary value problem

$$(BVP) \qquad -(a(x)u'(x))' = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0, \qquad a(x) > 0.$$

a) Formulate the variational formulation and minimization problem for BVP.
b) Show that the BVP and variational formulation are equivalent.
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Solutions.

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1. According to Lagrange interpolation theorem we have that

$$||f-\pi_1 f||_{L_\infty(0,1)} \le \frac{1}{2}(x-0)\cdot(1-x)\max_{x\in[0,1]}|f''|.$$

Further $g(x)=x(1-x)$ has a maximum for $g'(x)=0$, i.e. for $1\cdot(1-x)+x\cdot(-1)=0$, or $x=1/2$. Hence $\max_{x\in[0,1]}[x(1-x)]=\max_{x\in[0,1]}g(x)=1/2(1-1/2)=1/4$. which yields

$$||f-\pi_1 f||_{L_\infty(0,1)} \le \frac{1}{8}||f||_{L_\infty(0,1)}.$$

2. We multiply the differential equation by a test function $v\in H_0^1=\{v: ||v||+||v'||<\infty,\ v(0)=0\}$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u\in H_0^1(I)$ such that

$$\int_I (u'v'+u'v)=\int_I fv, \quad \forall v\in H_0^1(I). \tag{1}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U\in V_h^0$ such that

$$\int_I (U'v'+xU'v+Uv)=\int_I fv, \quad \forall v\in V_h^0\subset H_0^1(I), \tag{2}$$

where

$$V_h^0=\{v: v \text{ is piecewise linear and continuous in a partition of } I,\ v(0)=v(1)=0\}.$$

Now let $e=u-U$, then (??)-(??) gives that

$$\int_I (e'v'+xe'v+ev)=0, \quad \forall v\in V_h^0, \qquad \text{(Galerkin Ortogonalitet)}. \tag{3}$$

We note that using $e(0)=e(1)=0$, we get

$$\int_I xe'e=\frac{1}{2}\int_I x\frac{d}{dx}(e^2)=\frac{1}{2}(xe^2)|_0^1-\frac{1}{2}\int_I e^2=-\frac{1}{2}\int_I e^2, \tag{4}$$

Further, using Poincare inequality we have

$$\|e\|^2\le\|e'\|^2.$$

A priori error estimate: We use (??) and (??) to get

$$\begin{aligned}
\|e'\|^2_{L_2(I)}+\frac{1}{2}\|e\|^2_{L_2} &= \int_I (e'e'+\frac{1}{2}ee)=\int_I (e'e'+xe'e+ee)\\
&=\int_I \Big(e'(u-U)'+xe'(u-U)+e(u-U)\Big)=\{v=U-\pi_h u \text{ i}(??)\}\\
&=\int_I \Big(e'(u-\pi_h u)'+xe'(u-\pi_h u)+e(u-\pi_h u)\Big)\\
&\le \|(u-\pi_h u)'\|\|e'\|+\|u-\pi_h u\|\|e'\|+\|u-\pi_h u\|\|e\|\\
&\le \{\|(u-\pi_h u)'\|+\sqrt{2}\|u-\pi_h u\|\}\|e\|_{H^1}\\
&\le C_i\{\|hu''\|+\sqrt{2}\|h^2u''\|\}\|e\|_{H^1}.
\end{aligned}$$

this gives that

$$\|e\|_{H^1}\le 2C_i\{\|hu''\|+\sqrt{2}\|h^2u''\|\}.$$

which is the a priori error estimate.

A posteriori error estimate:

$$\|e'\|^2_{L_2(I)} + \frac{1}{2}\|e\|^2_{L_2} = \int_I (e'e' + \frac{1}{2}ee) = \int_I (e'e' + xe'e + ee)$$

$$= \int_I ((u-U)'e' + x(u-U)'e + (u-U)e) = \{v = e \ \text{ in } (??)\}$$

$$= \int_I fe - \int_I (U'e' + xU'e + Ue) = \{v = \pi_h e \ \text{ in } (??)\} \tag{5}$$

$$= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + xU'(e - \pi_h e) + U(e - \pi_h e)\Big)$$

$$= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),$$

where $\mathcal{R}(U) := f + U'' - xU' - U = f - xU' - U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$\|e'\|^2_{L_2(I)} + \frac{1}{2}\|e\|^2_{L_2} \le \|h\mathcal{R}(U)\|\|h^{-1}(e-\pi_h e)\| \le C_i\|h\mathcal{R}(U)\|\|e'\| \le \frac{1}{2}C_i^2\|h\mathcal{R}(U)\|^2 + \frac{1}{2}\|e'\|^2_{L_2(I)},$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

3. Let V be the linear function space defined by

$$V := \{v : v \ \text{ is continuous in } \Omega, \ v = 0, \ \text{ on } \partial\Omega\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$-(\Delta u, \nabla v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds = (\nabla u, \nabla v), \qquad \forall v \in V.$$

Thus the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega$:

$$V_h := \{v : v \ \text{ is continuous piecewise linear in } \Omega, \ v = 0, \ \text{ on } \partial\Omega\}.$$

The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{i=1}^3 \xi_i \varphi_i(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^{3} \xi_i \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \varphi_j \, dx \Big) = \int_\Omega f\varphi_j \, dx, \quad j = 1, 2, 3,$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1,x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1,x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\phi_2(x_1,x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\phi_3(x_1,x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1,\phi_1) = \int_T \phi_1^2\,dx = h^2\int_0^1\int_0^{1-x_2}(1-x_1-x_2)^2\,dx_1dx_2 = \frac{h^2}{12},$$
$$s_{11} = (\nabla\phi_1,\nabla\phi_1) = \int_T |\nabla\phi_1|^2\,dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1,\phi_1) = \int_T \phi_1^2\,dx = \frac{|T|}{3}\sum_{j=1}^{3}\phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0+\frac{1}{4}+\frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$M_{11} = M_{22} = M_{33} = 2m_{11} + 4m_{22} = \frac{1}{2}h^2, \qquad S_{11} = S_{22} = S_{33} = 2s_{11} + 4s_{22} = 4,$$
$$M_{12} = M_{23} = 2m_{12} = \frac{1}{12}h^2, \qquad S_{12} = S_{23} = 2s_{12} = -1,$$
$$M_{13} = 0 \qquad S_{13} = 0,$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 6 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 1 & 6 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

4. This is inspired from the proof of the Poincare inequality in the 1D case: We have, due to the vanishing boundary data, that

$$\begin{aligned}
|u(x)| = |u(x_1,x_2) - u(0,x_2)| &= \Big|\int_0^{x_1}\frac{\partial}{\partial\bar{x}_1}u(\bar{x}_1,x_2)\,d\bar{x}_1\Big| \\
&= \Big|\int_0^{x_1} 1\cdot\frac{\partial}{\partial\bar{x}_1}u(\bar{x}_1,x_2)\,d\bar{x}_1\Big| \le \{\text{Cauchy's inequality}\} \\
&\le \Big(\int_0^{x_1}1^2\,d\bar{x}_1\Big)^{1/2}\cdot\Big(\int_0^{x_1}(\frac{\partial}{\partial\bar{x}_1}u(\bar{x}_1,x_2))^2\,d\bar{x}_1\Big)^{1/2} \\
&\le \Big(\int_0^{1}(\frac{\partial}{\partial\bar{x}_1}u(\bar{x}_1,x_2))^2\,d\bar{x}_1\Big)^{1/2}.
\end{aligned}$$

This implies that

$$\int_\Omega |u|^2\,dx \le \int_\Omega \Big(\int_0^1 (\frac{\partial}{\partial \bar{x}_1} u(\bar{x}_1, x_2))^2\,d\bar{x}_1\Big)\,dx$$
$$= \int_0^1\int_0^1 \Big(\int_0^1 (\frac{\partial}{\partial \bar{x}_1} u(\bar{x}_1, x_2))^2\,d\bar{x}_1\Big)\,dx_1\,dx_2$$
$$= \int_0^1 \Big(\int_0^1 (\frac{\partial}{\partial \bar{x}_1} u(\bar{x}_1, x_2))^2\,d\bar{x}_1\Big)\,dx_2 = \int_0^1\int_0^1 (\frac{\partial}{\partial x_1} u(x_1, x_2))^2\,dx_1\,dx_2$$
$$= \int_\Omega (\frac{\partial}{\partial x_1} u(x_1, x_2))^2\,dx \le \int_\Omega |\nabla u|^2\,dx,$$

which gives the desired result:

$$\Big(\int_\Omega |u|^2\,dx\Big)^{1/2} \le \Big(\int_\Omega |\nabla u|^2\,dx\Big)^{1/2}.$$

5. See the Lecture Notes.

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