Mathematics Chalmers & GU

**TMA372/MMG800: Partial Differential Equations, 2011–06–04; kl 8.30-13.30.**

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 5p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p-     For GU students**G**:15-27p, **VG**: 28p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1011/index.html

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**1.** Let $\pi_1 f$ be the linear interpolant of a twice continuously differentiable function $f$ on the interval $(a, b)$. Prove that

$$||f - \pi_1 f||_{L_\infty(a,b)} \le (b-a)^2 ||f''||_{L_\infty(a,b)}.$$

**2.** Prove an a priori and an a posteriori error estimate for the cG(1) finite element method for

$$-u''(x) + u'(x) = f, \quad 0 < x < 1; \qquad u(0) = u(1) = 0.$$

**3.** Derive the cG(1)-cG(1), Crank-Nicolson approximation, for the initial boundary value problem

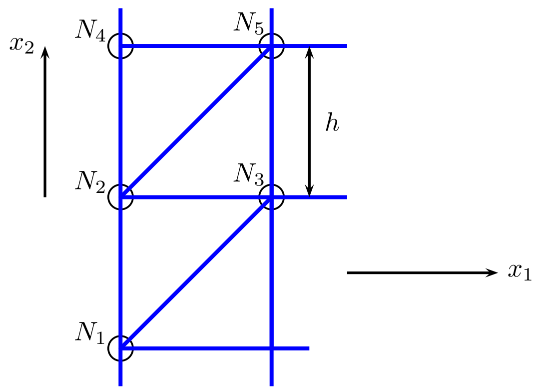
$$\begin{cases} \dot{u} - u'' = f, & 0 < x < 1, \quad t > 0, \\ u'(0,t) = u'(1,t) = 0, \quad u(x,0) = 0, & x \in [0,1], \quad t > 0, \end{cases} \tag{1}$$

**4.** Show that the cG(1)-cG(1) solution for wave equation in $1d$ satisfies the conservation of energy:

$$\|U_n'\| + \|\dot{U}_n\| = \|U_{n-1}'\| + \|\dot{U}_{n-1}\|. \tag{2}$$

**5.** Let $\Omega$ be the domain in the figure below, with the given triangulation and nodes $N_i$, $i = 1, \dots, 5$. Let $U$ be the cG(1) solution to the problem

$$-\Delta u = 1, \quad \text{in } \Omega \subset \mathbf{R}^2, \qquad \text{with} \qquad -\mathbf{n} \cdot \nabla u = 0, \quad \text{on } \partial\Omega. \tag{3}$$




a) Given the test function $\varphi_2$ at node $N_2$, find the relation between $U_1$, $U_2$, $U_3$, $U_4$, and $U_5$.
b) Derive the corresponding relation when the equation is replaced by $-\Delta u + (1,0) \cdot \nabla u = 1$.

**6.** (a) $p$ and $q$ are positive constants. Verify in details that the coefficient matrix for the $cG(1)$ method for

$$\begin{cases} -u''(x) + pu(x) = f(x), & x \in (0,1), \\ u'(0) = u'(1) = q, \end{cases}$$

is symmetric, positive definite and tridiagonal.
(b) For which values for the parameter $p$ is the coefficient matrix diagonal?

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void!

**TMA372/MMG800: Partial Differential Equations, 2011–06–04; kl 8.30-13.30..**
**Lösningar/Solutions.**

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**1.** See Lecture Notes.

**2.** We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over $I$. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + u'v) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{4}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$\int_I (U'v' + U'v) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{5}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I (e'v' + e'v) = 0, \quad \forall v \in V_h^0. \tag{6}$$

We note that using $e(0) = e(1) = 0$, we get

$$\int_I e'e = \int_I \frac{1}{2}\frac{d}{dx}\Big(e^2\Big) = \frac{1}{2}(e^2)|_0^1 = 0. \tag{7}$$

Further, using Poincare inequality we have

$$\|e\|^2 \le \|e'\|^2.$$

*A priori error estimate:* We use Poincare inequality and (7) to get

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \le 2\int_I e'e' = 2\int_I (e'e' + e'e) = 2\int_I \Big(e'(u-U)' + e'(u-U)\Big) \\
&= 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big) + 2\int_I \Big(e'(\pi_h u - U)' + e'(\pi_h u - U)\Big) \\
&= \{v = U - \pi_h u \ \text{ in (6)}\} = 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big) \\
&\le 2\|(u-\pi_h u)'\|\,\|e'\| + 2\|u-\pi_h u\|\,\|e'\| \\
&\le 2C_i\{\|hu''\| + \|h^2u''\|\}\|e\|_{H^1},
\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + \|h^2u''\|\},$$

which is the a priori error estimate.

*A posteriori error estimate:*

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \leq 2\int_I e'e' = 2\int_I (e'e' + e'e) \\
&= 2\int_I ((u-U)'e' + (u-U)'e) = \{v = e \ \text{ in (4)}\} \\
&= 2\int_I fe - \int_I (U'e' + U'e) = \{v = \pi_h e \ \text{ in (5)}\} \\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + U'(e - \pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned} \tag{8}$$

where $\mathcal{R}(U) := f + U'' - U' = f - U'$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$\begin{aligned}
\|e\|_{H^1}^2 &\leq \|h\mathcal{R}(U)\|\|h^{-1}(e - \pi_h e)\| \\
&\leq C_i \|h\mathcal{R}(U)\|\|e'\| \leq C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},
\end{aligned}$$

where $C_i$ is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \leq C_i \|h\mathcal{R}(U)\|.$$

**3.** Make the cG(1)-cG(1) ansatz

$$U(x,t) = U_{n-1}(x)\psi_{n-1}(t) + U_n(x)\psi_n(t), \qquad \text{with} \quad U_n(x) = \sum_{j=1}^{M} U_{n,j}\varphi_j(x),$$

in the variational formulation

$$\int_{I_n}\int_0^1 u'v' = \int_{I_n}\int_0^1 f\,v, \qquad I_n = (t_{n-1}, t_n).$$

Recall that $v = \varphi_j(x)$, $j = 1, \dots, M$ and

$$\psi_{n-1}(t) = \frac{t_n - t}{t_n - t_{n-1}}, \qquad \psi_n(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}.$$

For a uniform tile partition with $k := t_n - t_{n-1}$, this yields the equation system

$$(M + \frac{k}{2}S)U_n = (M - \frac{k}{2}S)U_{n-1} + k\mathbf{b}_n.$$

Here $U_n$ is the node-vale vector with entries $U_{n,j}$, $M$ is the mass-matrix with elements $\int_0^1 \varphi_i(x)\varphi_j(x)$, $S$ is the stiffness-matrix with elements $\int_0^1 \varphi_i'(x)\varphi_j'(x)$, and $\mathbf{b}_n$ is the load vector with elements $\frac{1}{k}\int_{I_n}\int_0^1 f\varphi_i(x)$. The corresponding dG0 ($\approx$ implicit Euler) time-stepping yields

$$(M + kS)U_n = MU_{n-1} + k\mathbf{b}_n.$$

**4.** Following the lecture notes, we may write the wave equation

$$\begin{cases}
\ddot{u} - u'' = 0, & 0 < x < 1, \qquad\qquad t > 0, \\
u(0,t) = 0, & u'(0,t) = g(t), t > 0, \\
u(x,0) = u_0(x), & \dot{u}(x,0) = v_0(x), \qquad 0 < x < 1,
\end{cases}$$

in a system viz,

$$\begin{cases}
\dot{u} = v, & t > 0, \\
\dot{v} = u'' & t > 0,
\end{cases}$$

for which the cG(1) method yields the matrix system

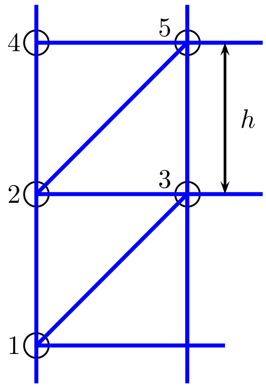
$$\begin{cases}
MU_n - \frac{k}{2}MV_n = MU_{n-1} + \frac{k}{2}MV_{n-1} \\
\frac{k}{2}SU_n + MV_n = -\frac{k}{2}SU_{n-1} + MV_{n-1} + g_n,
\end{cases}$$

with $M$ and $S$ being the mass and stiffness matrices, respectively. Let $g(t) \equiv 0$, and multiply the first equation by $(U_n + U_{n-1})^t SM^{-1}$ and the second equation by $(V_n + V_{n-1})^t$. Adding up and using the identities as $W_n^t S W_n = \|W_n'\|^2$, and $P^t AQ = Q^T AP$, for $A = S, M$ yields the desired result.

**5.** a) With $U$ expressed in terms of the basis functions $\varphi_j$, $j = 1, 2, 3, 4, 5$ and with the test function $v = \varphi_2$ in the variational formulation we obtain the relation

$$-\frac{1}{2}U_1 + 2U_2 - U_3 - \frac{1}{2}U_4 = \frac{1}{2}h^2.$$

b) If we change the equation to $-\Delta u + (1, 0) \cdot \nabla u = 1$ the relation between the nodal values




becomes:

$$-\frac{1}{2}U_1 + 2U_2 - U_3 - \frac{1}{2}U_4 - \frac{h}{3}U_2 + \frac{h}{3}U_3 - \frac{h}{6}U_4 + \frac{h}{6}U_5 = \frac{1}{2}h^2.$$

Finally if, for instance, for $-\nabla \cdot a\nabla u = f$ with $a = 1$ for $x < 0$ and $a = 2$ for $x_2 > 0$, the corresponding relation is:

$$-\frac{1}{2}U_1 + 3U_2 - \frac{3}{2}U_3 - U_4 = \frac{1}{2}h^2.$$

You may work out the details in such a model!

**6.** Let $V$ and $V_h$ be the spaces of continuous and discrete solutions, respectively. The variational formulation is: Find $u \in V$ such that

$$-\int_0^1 \Big(u''(x) - pu(x)\Big)v(x)\,dx = \int_0^1 f(x)v(x)\,dx, \qquad \forall v \in V.$$

Integrating by parts and using the fact that $u'(0) = u'(1) = q$ we get

$$-\int_0^1 u'v'\,dx + p\int_0^1 uv\,dx \int_0^1 f(x)v(x)\,dx + q(v(1) - v(0)), \qquad \forall v \in V.$$

The corresponding $cG(1)$ method reads: Find $U \in V_h$ such that

$$-\int_0^1 U'v'\,dx + p\int_0^1 Uv\,dx = \int_0^1 f(x)v(x)\,dx + q(v(1) - v(0)), \qquad \forall v \in V_h.$$

If $U = \sum_{j=1}^M \xi_j\varphi_j(x)$, then we get the following system of equations:

$$\int_0^1 \sum_{j=1}^M \xi_j\varphi_j'\varphi_i'\,dx + p\int_0^1 \sum_{j=1}^M \xi_j\varphi_j\varphi_i\,dx = \int_0^1 f(x)\varphi_i\,dx + q(\varphi_i(1) - \varphi_i(0)), \quad i = 1, \dots M.$$

Or equivalently

$$\sum_{j=1}^M \xi_j \int_0^1 \Big(\varphi_j'\varphi_i' + p\varphi_j\varphi_i\Big)\,dx = \int_0^1 f(x)\varphi_i\,dx + q(\varphi_i(1) - \varphi_i(0)), \quad i = 1, \dots M.$$

That is we have the system $A\xi = b$ with

$$\begin{cases} a_{ij} &= \int_0^1 \left(\varphi_j'\varphi_i' + p\varphi_j\varphi_i\right) dx = \tilde{a}_{ij} + p\int_0^1 \varphi_j\varphi_i\, dx, \qquad \tilde{a}_{ij} = \int_0^1 \varphi_j'\varphi_i'\, dx \\ b_i &= \int_0^1 f(x)\varphi_i\, dx + q(\varphi_i(1) - \varphi_i(0)). \end{cases}$$

The stiffness matrix is obviously *symmetric*, since $a_{ij} = a_{ji}$. To see if $A$ is positive definite, we form for any vector $v \in \mathbb{R}^M$

$$\begin{aligned} v^t Av &= \sum_{i=1}^M v_i\Big(\sum_{j=1}^M a_{ij}v_j\Big) = \sum_{i=1}^M v_i\Big[\sum_{j=1}^M v_j\int_0^1 \Big(\varphi_j'\varphi_i' + p\varphi_j\varphi_i\Big)\,dx\Big] \\ &= \int_0^1 \sum_{i=1}^M v_i\Big[\sum_{j=1}^M v_j\Big(\varphi_j'\varphi_i' + p\varphi_j\varphi_i\Big)\,dx\Big] \\ &= \int_0^1 \sum_{i=1}^M v_i\Big(\sum_{j=1}^M v_j\varphi_j'\varphi_i'\Big)\,dx + \int_0^1 \sum_{i=1}^M v_i\Big(\sum_{j=1}^M v_j p\varphi_j\varphi_i\Big)\,dx \qquad (9) \\ &= \int_0^1 \sum_{i=1}^M v_i\varphi_i'\Big(\sum_{j=1}^M v_j\varphi_j'\Big)\,dx + p\int_0^1 \sum_{i=1}^M v_i\varphi_i\Big(\sum_{j=1}^M v_j\varphi_j\Big)\,dx \\ &= \int_0^1 \Big(\sum_{i=1}^M v_i\varphi_i'\Big)^2\,dx + p\int_0^1 \Big(\sum_{i=1}^M v_i\varphi_i\Big)^2\,dx. \end{aligned}$$

Thus $v^t Av \geq 0$ and $v^t Av = 0 \iff v = 0$, since $p \geq 0$. Hence $A$ is positive definite.

To see if $A$ is tridiagonal we compute the elements $a_{ij}$:

$$a_{ij} = 0, \qquad \text{if} \quad |i - j| > 1. \qquad (10)$$

Since the support for the basis functions overlap only for adjacent nodes.

$$\begin{aligned} a_{ii} &= \tilde{a}_{ii} + p\int_0^1 \varphi_i^2\,dx \\ &= \frac{1}{h_i} + \frac{1}{h_{i+1}} + p\int_{x_{i-1}}^{x_i} \Big(\frac{x - x_{i-1}}{h_i}\Big)^2 dx + p\int_{x_i}^{x_{i+1}} \Big(\frac{x - x_{i+1}}{-h_{i+1}}\Big)^2 dx \qquad (11) \\ &= \frac{1}{h_i} + \frac{1}{h_{i+1}} + \frac{p}{3}(h_i + h_{i+1}). \end{aligned}$$

$$\begin{aligned} a_{i,i+1} &= \tilde{a}_{i,i+1} + p\int_0^1 \varphi_i\varphi_{i+1}\,dx \\ &= -\frac{1}{h_{i+1}} + p\int_{x_i}^{x_{i+1}} \frac{x - x_{i+1}}{-h_{i+1}} \cdot \frac{x - x_i}{h_{i+1}}\,dx \\ &= -\frac{1}{h_{i+1}} - \frac{p}{h_{i+1}^2}\Big[(x - x_{i+1})\frac{(x - x_i)^2}{2}\Big]_{x_i}^{x_{i+1}} + \frac{p}{h_{i+1}^2}\int_{x_i}^{x_{i+1}} \frac{(x - x_i)^2}{2} \qquad (12) \\ &= -\frac{1}{h_{i+1}} + \frac{p}{6}h_{i+1}. \end{aligned}$$

Obviously (10)-(12) means that $A$ is tridiagonal.

Since we may choose $p = \frac{6}{h_{i+1}^2}$, it is possible that $a_{i,i+1} = 0$. $A$ may even be diagonal (for a uniform triangulation). In general, though, $A$ is *tridiagonal*.

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