Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2010–08–25; kl 8.30-13.30.

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 7p. Valid bonus poits will be added to the scores.
Breakings: **3**: 20-29p, **4**: 30-39p och **5**: 40p- For GU **G** students :20-35p, **VG**: 36p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/0910/index.html

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1. Derive the cG(1)-cG(1), Crank-Nicolson approximation, for the initial boundary value problem

$$\begin{cases} \dot{u} - u'' = f, & 0 < x < 1, \quad t > 0, \\ u'(0,t) = u'(1,t) = 0, & u(x,0) = 0, \quad x \in [0,1], \ t > 0, \end{cases} \tag{1}$$

2. Show that the cG(1)-cG(1) solution for wave equation in $1d$ satisfies the conservation of energy:

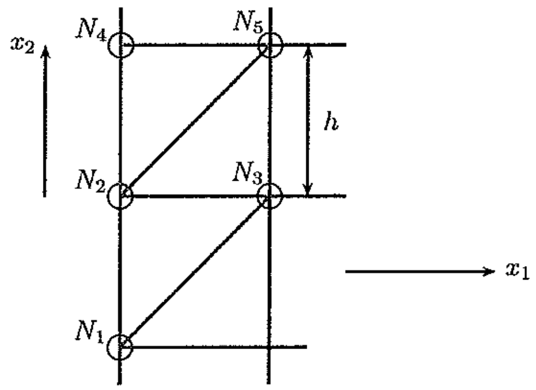
$$\|U_n'\| + \|\dot{U}_n\| = \|U_{n-1}'\| + \|\dot{U}_{n-1}\|. \tag{2}$$

3. Consider the Poisson equation with Neumann boundary condition

$$-\Delta u = f, \quad \text{in } \Omega \in \mathbf{R}^2, \qquad \text{with} \qquad -\mathbf{n}\cdot\nabla u = k\,u, \quad \text{on } \partial\Omega, \tag{3}$$

where $k > 0$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ ($\partial\Omega$ is the boundary of Ω).
a) Prove the Poincare inequality: $\|u\|_{L_2(\Omega)} \le C_\Omega(\|u\|_{L_2(\partial\Omega)} + \|\nabla u\|_{L_2(\Omega)})$.
b) Use the inequality in a) and show that $\|u\|_{L_2(\partial\Omega)} \to 0$ as $k \to \infty$.

4. Let Ω be the domain in the figure below, with the given triangulation and nodes N_i, $i = 1, \dots, 5$. Let U be the cG(1) solution to the problem (3), with $f = 1$ and $k = 0$.



a) Given the test function φ_2 at node N_2, find the relation between U_1, U_2, U_3, U_4, and U_5.
b) Derive the corresponding relation when the equation is replaced by $-\Delta u + (1,0)\cdot\nabla u = 1$.

5. a) Formulate a relevant minimization problem for the solution of the Poisson equation

$$-\Delta u = f, \quad \text{in } \Omega \in \mathbf{R}^2, \qquad \text{with} \quad \mathbf{n}\cdot\nabla u = b(g-u), \quad \text{on } \partial\Omega,$$

where $f > 0$, $b > 0$ and g are given functions.
b) Derive an *a priori* error estimate for cG(1) approximation in the corresponding energy-norm.

6. Formulate and prove the Lax-Milgram theorem.

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Lösningar/Solutions.

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1. Make the cG(1)-cG(1) ansatz

$$U(x,t) = U_{n-1}(x)\psi_{n-1}(t) + U_n(x)\psi_n(t), \qquad \text{with} \quad U_n(x) = \sum_{j=1}^{M} U_{n,j}\varphi_j(x),$$

in the variational formulation

$$\int_{I_n}\int_0^1 u'v' = \int_{I_n}\int_0^1 f\,v, \qquad I_n = (t_{n-1}, t_n).$$

Recall that $v = \varphi_j(x)$, $j = 1, \dots, M$ and

$$\psi_{n-1}(t) = \frac{t_n - t}{t_n - t_{n-1}}, \qquad \psi_n(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}.$$

For a uniform tile partition with $k := t_n - t_{n-1}$, this yields the equation system

$$(M + \frac{k}{2}S)U_n = (M - \frac{k}{2}S)U_{n-1} + k\mathbf{b}_n.$$

Here U_n is the node-vale vector with entries $U_{n,j}$, M is the mass-matrix with elements $\int_0^1 \varphi_i(x)\varphi_j(x)$, S is the stiffness-matrix with elements $\int_0^1 \varphi_i'(x)\varphi_j'(x)$, and $\mathbf{b}_n$ is the load vector with elements $\frac{1}{k}\int_{I_n}\int_0^1 f\varphi_i(x)$. The corresponding dG0 ($\approx$ implicit Euler) time-stepping yields

$$(M + kS)U_n = MU_{n-1} + k\mathbf{b}_n.$$

2. Following the lecture notes, we may write the wave equation

$$\begin{cases} \ddot{u} - u'' = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, & u'(0,t) = g(t), t > 0, \\ u(x,0) = u_0(x), & \dot{u}(x,0) = v_0(x), \quad 0 < x < 1, \end{cases}$$

in a system viz,

$$\begin{cases} \dot{u} = v, & t > 0, \\ \dot{v} = u'' & t > 0, \end{cases}$$

for which the cG(1) method yields the matrix system

$$\begin{cases} MU_n - \frac{k}{2}MV_n = MU_{n-1} + \frac{k}{2}MV_{n-1} \\ \frac{k}{2}SU_n + MV_n = -\frac{k}{2}SU_{n-1} + MV_{n-1} + g_n, \end{cases}$$

with M and S being the mass and stiffness matrices, respectively. Let $g(t) \equiv 0$, and multiply the first equation by $(U_n + U_{n-1})^t SM^{-1}$ and the second equation by $(V_n + V_{n-1})^t$. Adding up and using the identities as $W_n^t SW_n = \|W_n'\|^2$, and $P^tAQ = Q^TAP$, for $A = S, M$ yields the desired result.

3. a) There is smooth function ϕ such that $\Delta\phi = 1$ so that, using Greens formula

$$\begin{aligned}\|u\|_\Omega^2 &= \int_\Omega u^2\,\Delta\phi = \int_{\partial\Omega} u^2\,\partial_n\phi - \int_\Omega 2u\nabla u\cdot\nabla\phi \\ &\le C_1\|u\|_{\partial\Omega}^2 + C_2\|u\|\|\nabla u\| \le C_1\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|u\|_\Omega^2 + \frac{1}{2}C_2^2\|\nabla u\|_\Omega^2.\end{aligned}$$

This yields

$$\|u\|_\Omega^2 \le 2C_1\|u\|_{\partial\Omega}^2 + C_2^2\|\nabla u\|_\Omega^2 \le C^2(\|u\|_{\partial\Omega}^2 + \|\nabla u\|_\Omega^2),$$

where $C^2 = \max(2C_1, C_2^2)$, $C_1 = \max_{\partial\Omega}|\partial_n\phi|$, and $C_2 = \max_\Omega(2|\nabla\phi|)$.

b) Multiply the equation $-\Delta u = f$ by u and integrate over Ω. Partial integration together with the boundary data $-\partial_n u = ku$ and Cauchy's inequality, yields

$$\begin{aligned}\|\nabla u\|_\Omega^2 + k\|u\|_{\partial\Omega}^2 &= \int_\Omega \nabla u\cdot\nabla u + \int_{\partial\Omega} u(-\partial_n u) = \int_\Omega u(-\Delta u) = \int_\Omega fu \\ &\le \|u\|_\Omega^{\|} f\|_\Omega \le C_\Omega(\|u\|_{\partial\Omega} + \|\nabla u\|_\Omega)\|f\|_\Omega = \|u\|_{\partial\Omega}C_\Omega\|f\|_\Omega + \|\nabla u\|_\Omega C_\Omega\|f\|_\Omega \\ &\le \frac{1}{2}\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|\nabla u\|_\Omega^2 + C_\Omega^2\|f\|_\Omega^2.\end{aligned}$$

Subtracting $\frac{1}{2}\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|\nabla u\|_\Omega^2$ from the both sides, we end up with

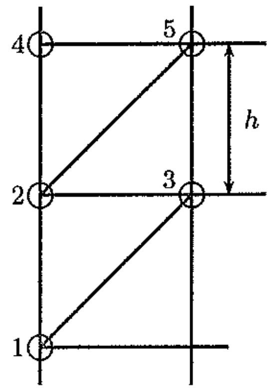
$$(k - \frac{1}{2})\|u\|_{\partial\Omega}^2 \le \frac{1}{2}\|\nabla u\|_\Omega^2 + (k - \frac{1}{2})\|u\|_{\partial\Omega}^2 \le C_\Omega^2\|f\|_\Omega^2,$$

which gives that $\|u\|_{\partial\Omega} \to 0$ as $k \to \infty$.

4. a) With U expressed in terms of the basis functions φ_j, $j = 1,2,3,4,5$ and with the test function $v = \varphi_2$ in the variational formulation we obtain the relation

$$-\frac{1}{2}U_1 + 2U_2 - U_3 - \frac{1}{2}U_4 = \frac{1}{2}h^2.$$

b) If we change the equation to $-\Delta u + (1,0)\cdot\nabla u = 1$ the relation between the nodal values



becomes:

$$-\frac{1}{2}U_1 + 2U_2 - U_3 - \frac{1}{2}U_4 - \frac{h}{3}U_2 + \frac{h}{3}U_3 - \frac{h}{6}U_4 + \frac{h}{6}U_5 = \frac{1}{2}h^2.$$

Finally if, for instance, for $-\nabla\cdot a\nabla u = f$ with $a = 1$ for $x < 0$ and $a = 2$ for $x_2 > 0$, the corresponding relation is:

$$-\frac{1}{2}U_1 + 3U_2 - \frac{3}{2}U_3 - U_4 = \frac{1}{2}h^2.$$

You may work out the details in such a model!

5. a) Multiply the equation by v, integrate over Ω, partial integrate, and use the boundary data to obtain

$$\int_\Omega fv = -\int_\Omega (\Delta u)v = -\int_\Gamma (\mathbf{n}\cdot\nabla u)v + \int_\Omega \nabla u\cdot\nabla v = \int_\Gamma buv - \int_\Gamma bgv + \int_\Omega \nabla u \nabla v,$$

where $\Gamma := \partial\Omega$. This can be rewritten as

$$\underbrace{\int_\Omega \nabla u\cdot\nabla v + \int_\Gamma buv}_{:=a(u,v)} = \underbrace{\int_\Omega fv + \int_\Gamma bgv}_{:=l(v)}.$$

Let now

$$F(w) = \frac{1}{2} = a(w,w) - l(w) = \frac{1}{2}\int_\Omega \nabla w\cdot\nabla w + \int_\Gamma bww - \int_\Omega fv + \int_\Gamma bgv,$$

and choose $w = u + v$, then

$$F(w) = F(u+v) = F(u) +$$
$$+\underbrace{\int_\Omega \nabla u\cdot\nabla v + \int_\Gamma buv - \int_\Omega fv + \int_\Gamma bgv}_{=0} + \underbrace{\frac{1}{2}\int_\Omega \nabla v\cdot\nabla v + \frac{1}{2}\int_\Gamma bvv}_{\geq 0} \geq F(u).$$

This gives $F(u) \leq F(w)$ for arbitrary w.

b) Make the discrete ansatz $U = \sum_{j=1}^{M} U_j\varphi_j$, and set $v = \varphi_i$, $i = 1, 2, \ldots, M$ in the variational formulation. Then we get the equation system $AU = B$, where U is the column vector with entries U_j, B is the load vector with elements

$$B_j = \int_\Omega f\varphi_i + \int_\Gamma bg\varphi_i,$$

and A is the matrix with elements

$$A_{ij} = \int_\Omega \nabla\varphi_i\cdot\nabla\varphi_j + \int_\Gamma b\varphi_i\varphi_j.$$

Here $\varphi_j = \varphi_j(x)$ is the basis function (*hat-functions*) for the set of all piecewise linear polynomials functions on a triangulation of the domain Ω.

Finally for the energy-norm $\|v\| = a(v,v)^{1/2}$, using the definition for $U = U(x)$, and the Galerkin orthogonality, we estimate the error $e = u - U$ as

$$\|e\|^2 = a(e,e) = a(e, u-U) = a(e,u) - a(e,U) = a(e,u)$$
$$= a(e,u) - a(e,v) = a(e, u-v) \leq \|e\|\|u-v\|.$$

This gives $\|u-U\| = \|e\| \leq \|u-v\|$, for arbitrary piecewise linear function v, due to the fact that for such U and v Galerkin orthogonality gives $a(e,U) = 0$ and $a(e,v) = 0$: Just notice that both U and v are the linear combination of the basis functions φ_j for which according to the definition of U we have that

$$a(e,\varphi_j) = a(u,\varphi_j) - a(U,\varphi_j) = l(\varphi_j) - l(\varphi_j) = 0.$$

In particular, we may chose the piecewise linear function v to be the interpolant u and hence get

$$\|u-U\| \leq \|u-v\| \leq C\|hD^2u\|,$$

where h is the mesh size and C is an interpolation constant independent of h and u.

6. See Lecture Notes or text book chapter 21.

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