

**TMA372/MAN660 Partial Differential Equations TM**

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

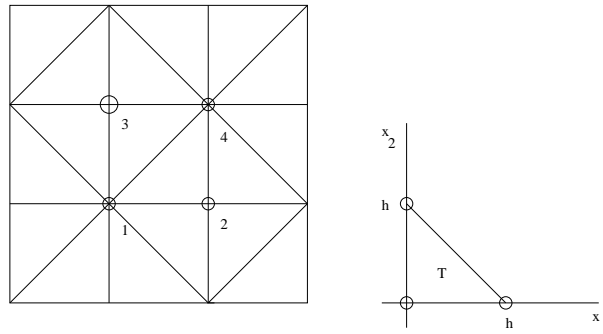
1. Let  $\alpha$  and  $\beta$  be positive constants. Give the piecewise linear finite element approximation procedure, on the uniform mesh, for the problem

$$-u''(x) = 1, \quad 0 < x < 1; \quad u(0) = \alpha, \quad u'(1) = \beta.$$

2. Formulate the  $cG(1)$  method for the boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega.$$

Write down the matrix form of the resulting equation system using the following uniform mesh:



3. Consider the boundary value problem for the stationary heat flow in  $1D$ :

$$(BVP) \quad -(a(x)u'(x))' = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

Formulate the corresponding variational formulation ( $VF$ ), minimization problem ( $MP$ ) and show that:  $(BVP) \iff (VF) \iff (MP)$ .

4. (a) Prove an a priori and an a posteriori error estimate for the finite element method for the problem

$$-u''(x) + u'(x) = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

- (b) Give an adaptive algorithm based on a posteriori estimates.

5. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$ . Consider the initial-boundary value problem

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases}$$

Show the stability estimates

$$(i) \quad \|\nabla u(t)\|^2 \leq \frac{1}{2t} \|v\|^2, \quad \text{and} \quad (ii) \quad \int_0^t s \|\Delta u(s)\|^2 ds \leq \frac{1}{4} \|v\|^2.$$

**TMA371 Partial Differential Equations TM, 2004-12-14. Solutions**

1. Let  $\alpha$  and  $\beta$  be positive constants. Give the piecewise linear finite element approximation procedure, on the uniform mesh, for the problem

$$-u''(x) = 1, \quad 0 < x < 1; \quad u(0) = \alpha, \quad u'(1) = \beta.$$

**Solution:** Multiply the pde by a test function  $v$  with  $v(0) = 0$ , integrate over  $x \in (0, 1)$  and use partial integration to get

$$\begin{aligned} & -[u'v]_0^1 + \int_0^1 u'v' dx = \int_0^1 v dx \quad \iff \\ (1) \quad & -u'(1)v(1) + u'(0)v(0) + \int_0^1 u'v' dx = \int_0^1 v dx \quad \iff \\ & -\beta v(1) + \int_0^1 u'v' dx = \int_0^1 v dx. \end{aligned}$$

The continuous variational formulation is now formulated as follows: Find

$$(VF) \quad u \in V := \{w : \int_0^1 (w(x)^2 + w'(x)^2) dx < \infty, \quad w(0) = \alpha\},$$

such that

$$\int_0^1 u'v' dx = \int_0^1 v dx + \beta v(1), \quad \forall v \in V^0,$$

where

$$V^0 := \{v : \int_0^1 (v(x)^2 + v'(x)^2) dx < \infty, \quad v(0) = 0\}.$$

For the discrete version we let  $\mathcal{T}_h$  be a uniform partition:  $0 = x_0 < x_1 < \dots < x_{M+1}$  of  $[0, 1]$  into the subintervals  $I_n = [x_{n-1}, x_n]$ ,  $n = 1, \dots, M + 1$ . Here, we have  $M$  interior nodes:  $x_1, \dots, x_M$ , two boundary points:  $x_0 = 0$  and  $x_{M+1} = 1$  and hence  $M + 1$  intervals.

The finite element method (discrete variational formulation) is now formulated as follows: Find

$$(FEM) \quad U \in V_h := \{w_h : w_h \text{ is piecewise linear, continuous on } \mathcal{T}_h, w_h(0) = \alpha\},$$

such that

$$(2) \quad \int_0^1 U'v_h' dx = \int_0^1 v_h dx + \beta v_h(1), \quad \forall v \in V_h^0,$$

where

$$V_h^0 := \{v_h : v_h \text{ is piecewise linear, continuous on } \mathcal{T}_h, v_h(0) = 0\}.$$

Using the basis functions  $\varphi_j$ ,  $j = 0, \dots, M + 1$ , where  $\varphi_1, \dots, \varphi_M$  are the usual *hat-functions* whereas  $\varphi_0$  and  $\varphi_{M+1}$  are *semi-hat-functions* viz;

$$(3) \quad \varphi_j(x) = \begin{cases} 0, & x \notin [x_{j-1}, x_j] \\ \frac{x-x_{j-1}}{h} & x_{j-1} \leq x \leq x_j \\ \frac{x_{j+1}-x}{h} & x_j \leq x \leq x_{j+1} \end{cases}, \quad j = 1, \dots, M.$$

and

$$\varphi_0(x) = \begin{cases} \frac{x_1-x}{h} & 0 \leq x \leq x_1 \\ 0, & x_1 \leq x \leq 1 \end{cases}, \quad \varphi_{M+1}(x) = \begin{cases} \frac{x-x_M}{h} & x_M \leq x \leq x_{M+1} \\ 0, & 0 \leq x \leq x_M. \end{cases}$$

In this way we may write

$$V_h = \alpha\varphi_0 \oplus [\varphi_1, \dots, \varphi_{M+1}], \quad V_h^0 = [\varphi_1, \dots, \varphi_{M+1}].$$

Thus every  $U \in V_h$  can be written as  $U = \alpha\varphi_0 + v_h$  where  $v_h \in V_h^0$ , i.e.,

$$U = \alpha\varphi_0 + \xi_1\varphi_1 + \dots + \xi_{M+1}\varphi_{M+1} = \alpha\varphi_0 + \sum_{i=1}^{M+1} \xi_i\varphi_i \equiv \alpha\varphi_0 + \tilde{U},$$

where  $\tilde{U} \in V_h^0$ , and hence the problem (2) can equivalently be formulated as to find  $\xi_1, \dots, \xi_{M+1}$  such that

$$\int_0^1 \left( \alpha\varphi_0' + \sum_{i=1}^{M+1} \xi_i\varphi_i' \right) \varphi_j' dx = \int_0^1 \varphi_j dx + \beta\varphi_j(1), \quad j = 1, \dots, M+1,$$

which can be written as

$$\sum_{i=1}^{M+1} \left( \int_0^1 \varphi_j' \varphi_i' dx \right) \xi_i = - \int_0^1 \varphi_0' \varphi_j' dx + \int_0^1 \varphi_j dx + \beta\varphi_j(1), \quad j = 1, \dots, M+1,$$

or equivalently  $A\xi = b$  where  $A = (a_{ij})$  is the tridiagonal matrix with entries

$$a_{ii} = 2, \quad a_{i,i+1} = a_{i+1,i} = -1, \quad i = 1, \dots, M, \quad \text{and} \quad a_{M+1,M+1} = 1,$$

i.e.,

$$A = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ \dots & & & & & & \\ \dots & & & & & & \\ \dots & & & & & & \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 1 \end{bmatrix},$$

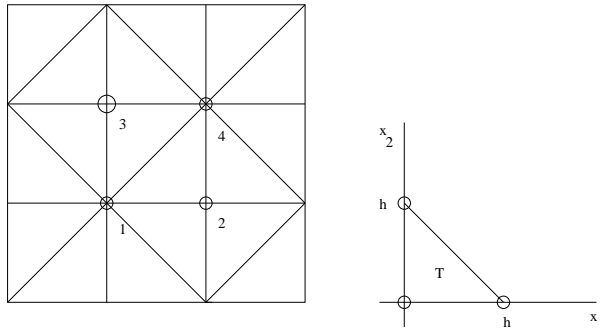
and the unknown  $\xi$  and the data  $b$  are given by

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_M \\ \xi_{M+1} \end{bmatrix}, \quad b = \begin{bmatrix} \int_0^1 \varphi_1 dx - \alpha \int_0^1 \varphi_0' \varphi_1' dx \\ \int_0^1 \varphi_2 dx \\ \vdots \\ \int_0^1 \varphi_M dx \\ \int_0^1 \varphi_{M+1} dx + \beta\varphi_{M+1}(1) \end{bmatrix} = \begin{bmatrix} h + \frac{1}{h}\alpha \\ h \\ \vdots \\ h \\ \frac{h}{2} + \beta \end{bmatrix}.$$

2. Formulate the  $cG(1)$  method for the boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega.$$

Write down the matrix form of the resulting equation system using the following uniform mesh:



**Solution:** Let  $V_h$  be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition  $v = 0$  on  $\partial\Omega$ . The  $cG(1)$  method is: Find  $U \in V_h$  such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \quad \forall v \in V_h$$

Making the “Ansatz”  $U(x) = \sum_{i=1}^4 \xi_i \varphi_i(x)$ , where  $\varphi_i$  are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^4 \xi_i \left( \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j dx + \int_{\Omega} \varphi_i \varphi_j dx \right) = \int_{\Omega} f \varphi_j dx, \quad j = 1, \dots, 4,$$

or, in matrix form,

$$(S + M)\xi = F,$$

where  $S_{ij} = (\nabla \varphi_i, \nabla \varphi_j)$  is the stiffness matrix,  $M_{ij} = (\varphi_i, \varphi_j)$  is the mass matrix, and  $F_j = (f, \varphi_j)$  is the load vector.

We first compute the mass and stiffness matrix for the reference triangle  $T$ . The local basis functions are

$$\begin{aligned} \phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla \phi_1(x_1, x_2) &= -\frac{1}{h} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla \phi_2(x_1, x_2) &= \frac{1}{h} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla \phi_3(x_1, x_2) &= \frac{1}{h} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence, with  $|T| = \int_T dz = h^2/2$ ,

$$\begin{aligned} m_{11} &= (\phi_1, \phi_1) = \int_T \phi_1^2 dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 dx_1 dx_2 = \frac{h^2}{12}, \\ s_{11} &= (\nabla \phi_1, \nabla \phi_1) = \int_T |\nabla \phi_1|^2 dx = \frac{2}{h^2} |T| = 1. \end{aligned}$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 dx = \frac{|T|}{3} \sum_{j=1}^3 \phi_1(\hat{x}_j)^2 = \frac{h^2}{6} \left(0 + \frac{1}{4} + \frac{1}{4}\right) = \frac{h^2}{12},$$

where  $\hat{x}_j$  are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad s = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices  $M$  and  $S$  from the local ones  $m$  and  $s$ :

$$\begin{aligned} M_{11} = M_{44} = 8m_{22} &= \frac{8}{12}h^2, & S_{11} = S_{44} = 8s_{22} &= 4, \\ M_{12} = M_{13} = M_{24} = M_{34} = 2m_{12} &= \frac{1}{12}h^2, & S_{12} = S_{13} = S_{24} = S_{34} = 2s_{12} &= -1, \\ M_{14} = 2m_{23} &= \frac{1}{12}h^2, & S_{14} = 2s_{23} &= 0, \\ M_{22} = M_{33} = 4m_{11} &= \frac{4}{12}h^2, & S_{22} = S_{33} = 4s_{11} &= 4, \\ M_{23} &= 0, & S_{23} &= 0. \end{aligned}$$

The remaining matrix elements are obtained by symmetry  $M_{ij} = M_{ji}$ ,  $S_{ij} = S_{ji}$ . Hence,

$$M = \frac{h^2}{12} \begin{bmatrix} 8 & 1 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix}, \quad S = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

**3.** Consider the boundary value problem for the stationary heat flow in  $1D$ :

$$(BVP) \quad -(a(x)u'(x))' = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

Formulate the corresponding variational formulation (VF), minimization problem (MP) and show that:  $(BVP) \iff (VF) \iff (MP)$ .

**Solution:** See PDE Lecture Notes, Chapter 8.

4. (a) Prove an a priori and an a posteriori error estimate for the finite element method for the problem

$$-u''(x) + u'(x) = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

(b) Give an adaptive algorithm based on a posteriori estimates.

**Solution:** (a) We multiply the differential equation by a test function  $v \in H_0^1(I)$ ,  $I = (0, 1)$  and integrate over  $I$ . Using partial integration and the boundary conditions we get the following *variational problem*: Find  $u \in H_0^1(I)$  such that

$$(4) \quad \int_I (u'v' + u'v) = \int_I f v, \quad \forall v \in H_0^1(I).$$

A *Finite Element Method* with  $cG(1)$  reads as follows: Find  $U \in V_h^0$  such that

$$(5) \quad \int_I (U'v' + U'v) = \int_I f v, \quad \forall v \in V_h^0 \subset H_0^1(I),$$

where

$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I, v(0) = v(1) = 0\}$ .

Now let  $e = u - U$ , then (1)-(2) gives that

$$(6) \quad \int_I (e'v' + e'v) = 0, \quad \forall v \in V_h^0.$$

We note that using  $e(0) = e(1) = 0$ , we get

$$(7) \quad \int_I e'e = \int_I \frac{1}{2} \frac{d}{dx} (e^2) = \frac{1}{2} (e^2)|_0^1 = 0.$$

Further, using Poicare inequality we have

$$\|e\|^2 \leq \|e'\|^2.$$

*A priori error estimate:* We use Poicare inequality and (7) to get

$$\begin{aligned} \|e\|_{H^1}^2 &= \int_I (e'e' + ee) \leq 2 \int_I e'e' = 2 \int_I (e'e' + e'e) = 2 \int_I (e'(u - U)' + e'(u - U)) \\ &= 2 \int_I (e'(u - \pi_h u)' + e'(u - \pi_h u)) + 2 \int_I (e'(\pi_h u - U)' + e'(\pi_h u - U)) \\ &= \{v = U - \pi_h u \text{ in (6)}\} = 2 \int_I (e'(u - \pi_h u)' + e'(u - \pi_h u)) \\ &\leq 2 \|(u - \pi_h u)'\| \|e'\| + 2 \|u - \pi_h u\| \|e'\| \\ &\leq 2C_i \{\|hu''\| + \|h^2 u''\|\} \|e\|_{H^1}, \end{aligned}$$

this gives that

$$\|e\|_{H^1} \leq C_i \{\|hu''\| + \|h^2 u''\|\},$$

which is the a priori error estimate.

A posteriori error estimate:

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \leq 2 \int_I e'e' = 2 \int_I (e'e' + e'e) \\
&= 2 \int_I ((u-U)'e' + (u-U)'e) = \{v = e \text{ in (4)}\} \\
(8) \quad &= 2 \int_I fe - \int_I (U'e' + U'e) = \{v = \pi_h e \text{ in (5)}\} \\
&= \int_I f(e - \pi_h e) - \int_I (U'(e - \pi_h e)' + U'(e - \pi_h e)) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned}$$

where  $\mathcal{R}(U) := f + U'' - U' = f - U'$ , (for approximation with piecewise linears,  $U \equiv 0$ , on each subinterval). Thus (5) implies that

$$\begin{aligned}
\|e\|_{H^1}^2 &\leq \|h\mathcal{R}(U)\| \|h^{-1}(e - \pi_h e)\| \\
&\leq C_i \|h\mathcal{R}(U)\| \|e'\| \leq C_i \|h\mathcal{R}(U)\| \|e\|_{H^1},
\end{aligned}$$

where  $C_i$  is an interpolation constant, and hence we have with  $\|\cdot\| = \|\cdot\|_{L_2(I)}$  that

$$\|e\|_{H^1} \leq C_i \|h\mathcal{R}(U)\|.$$

(b) An adaptive procedure can be formulated in the following steps:

Step I. Start with a given mesh size  $h$  and a given error tolerance “TOL”. Compute  $U$  and  $\mathcal{R}(U)$  corresponding to this  $h$

Step II. Compare  $C_i \|h\mathcal{R}(U)\|$  with the tolerance “TOL”:

IIa). If

$$C_i \|h\mathcal{R}(U)\| < TOL,$$

then accept  $U$  as an appropriate cG(1) approximate solution.

IIb). If

$$C_i \|h\mathcal{R}(U)\| \geq TOL,$$

then refine the mesh on the subintervals with large  $\mathcal{R}(U)$  contributions, thus obtain a new mesh and return to Step I.

5. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$ . Consider the initial-boundary value problem

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases}$$

Show the stability estimates

$$(i) \quad \|\nabla u(t)\|^2 \leq \frac{1}{2t} \|v\|^2, \quad \text{and} \quad (ii) \quad \int_0^t s \|\Delta u(s)\|^2 ds \leq \frac{1}{4} \|v\|^2.$$

**Solution:** See PDE Lecture Notes, Chapter 15.

MA