KVANTROV 2

Anders

III 7

$\lambda_{blå} = 400$ nm

$\lambda_e = \frac{h}{p}$; $\underbrace{p = m_e v, \text{ och } E_k = \frac{m_e v^2}{2}}_{\text{icke-rel.}} \Leftrightarrow v = \sqrt{\frac{2E_k}{m_e}}$

Koll, icke-rel:

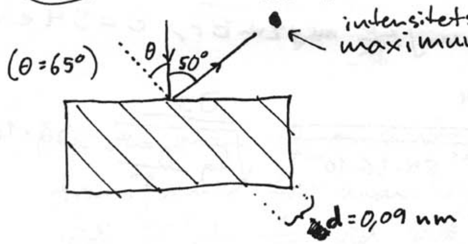
$V = 10$ kV $\Rightarrow E_k = eV = 10$ keV $\Rightarrow v = 140$ m/s $\ll c \Rightarrow$ OK!

Relativ upplösning: $\frac{a/\lambda_e}{a/\lambda_{blå}} = \frac{\lambda_{blå}}{\lambda_e} = \frac{\lambda_{blå}\, p}{h} =$

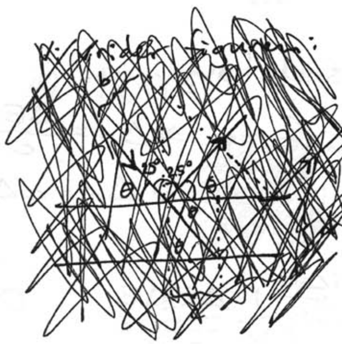
$= \frac{\lambda_{blå}\sqrt{2E_k}\cdot m_e}{h\sqrt{m_e}} = \frac{\lambda_{blå}\sqrt{2 m_e E_k}}{h} = 30\,000$

(FACIT: 300 000...)

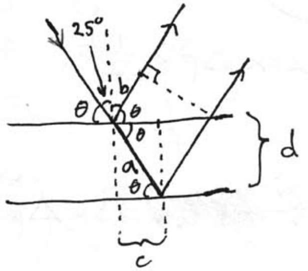
III 16 Braggspridning



Sökt: λ

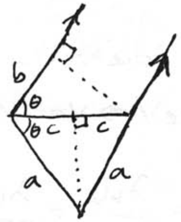


Vi vrider figuren:



...

... För att få maximum ska vågtopparna för de två strålarna sammanfalla, dvs skillnaden Δ i deras våglängd ska vara ett helt antal våglängder, $\Delta = n\lambda$. Vi har $\boxed{\Delta = 2a - b}$



Trigonometri: $a = \frac{d}{\sin\theta}$, $c = \frac{d}{\tan\theta}$

$b = 2c\cos\theta = \frac{2d\cos\theta}{\tan\theta}$

$\Rightarrow \Delta = \frac{2d}{\sin\theta} - \frac{2d\cos^2\theta}{\sin\theta} = 2d\,\frac{1-\cos^2\theta}{\sin\theta} = 2d\sin\theta$

$\Rightarrow 2d\sin\theta = n\lambda$

a) $\lambda = 2\cdot 0{,}09\cdot 10^{-9}\cdot \sin 65°\,\text{m} = 1{,}63$ Å

b) de Broglie: $\nu = \frac{v}{\lambda} = \frac{E}{h}$

$\lambda = \frac{h}{p}$, $p = mv = m\sqrt{\frac{2E}{m}} = \sqrt{2mE}$, $E = 54$ eV

$\Rightarrow \lambda = \frac{6{,}6\cdot 10^{-34}}{\sqrt{2\cdot 9{,}1\cdot 10^{-31}\cdot 54\cdot 1{,}6\cdot 10^{-19}}} \; \frac{\text{Js}}{\sqrt{\text{kg}\,\frac{\text{kgm}^2}{\text{s}^2}}} = 1{,}6\cdot 10^{-10}$ m

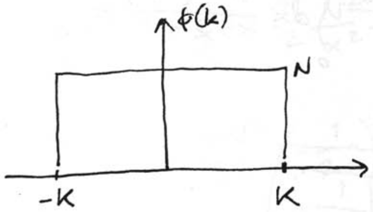
OK!

III 22 Heisenbergs osäkerhetsrelation

$\Delta x \Delta p \geq \frac{1}{2}\hbar$. För foton: $\begin{cases} E_\gamma = pc \Rightarrow \Delta p = \frac{\Delta E}{c} \\ x = ct \Rightarrow \Delta x = c\Delta t \end{cases}$

$\Rightarrow \Delta x \Delta p = c\Delta t \frac{\Delta E}{c} = \Delta t \Delta E$ OK!

III 25 $\quad \phi(k) = N\left(\theta(k+K) - \theta(k-K)\right)$



(invers) $\mathcal{F}$-transform mellan x- & k-rummet.

För att få tillbaka ursprunglig fkn vid $\mathcal{F}^{-1} \circ \mathcal{F}$ krävs en total faktor $\frac{1}{2\pi}$ till transformintegralerna. BETA har $\frac{1}{2\pi}$ vid $\mathcal{F}^{-1}$ och bara 1 vid $\mathcal{F}$. För att kunna normera både $\psi(x)$ och $\phi(k)$ samtidigt sätter vi istället $\frac{1}{\sqrt{2\pi}}$ vid både $\mathcal{F}$ & $\mathcal{F}^{-1}$.

Alltså: $\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \phi(k) e^{ikx} = \mathcal{F}^{-1}[\phi(k)]$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \psi(x) e^{-ikx} = \mathcal{F}[\psi(k)]$$

(Så om man vill använda BETA för att slå upp transformer får man lägga till en faktor $\sqrt{2\pi}$ vid $\mathcal{F}^{-1}$ och $\frac{1}{\sqrt{2\pi}}$ vid $\mathcal{F}$.)

a) $\psi(x) = \mathcal{F}^{-1}[\phi(k)] = \frac{1}{\sqrt{2\pi}} \int_{-K}^{K} dk\, N e^{ikx} =$

$$= \frac{N}{\sqrt{2\pi}} \frac{1}{ix} \left[e^{ikx}\right]_{-K}^{K} =$$

$$= \frac{2N}{\sqrt{2\pi}} \frac{1}{x} \left(\frac{e^{iKx} - e^{-iKx}}{2i}\right) = \sqrt{\frac{2}{\pi}} N \frac{\sin Kx}{x}$$

(OBS: skiljer sig från facit med $\frac{1}{\sqrt{2\pi}}$) ...

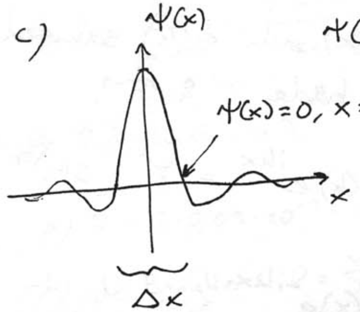
... b) Normalisering: $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

dvs $1 = \frac{2N^2}{\pi} \int_{-\infty}^{\infty} dx \frac{\sin^2(Kx)}{x^2} \underset{\text{jämn fkn}}{=} \frac{4N^2}{\pi} \int_{0}^{\infty} dx \frac{\sin^2(Kx)}{x^2} =$

$$= \frac{\cancel{4}^{\,2} N^2}{\cancel{\pi}} \frac{\cancel{\pi} K}{\cancel{2}} \iff \boxed{N = \sqrt{\frac{1}{2K}}}$$

$$\Rightarrow \psi(x) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{2K}} \frac{\sin Kx}{x} = \frac{1}{\sqrt{\pi K}} \frac{\sin Kx}{x}$$

c)

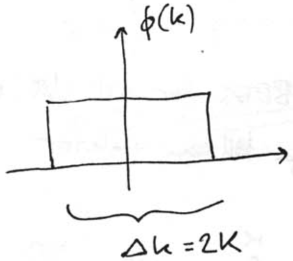


$\psi(x) = 0 \iff \sin Kx = 0 \Rightarrow Kx = \pi$

$\iff \underline{x = \pi/K}$

$\therefore\ \Delta x = \frac{2\pi}{K},\ \Delta k = 2K$

$\Delta p = \Delta(\hbar k) = \hbar \Delta k = \hbar 2K$



$\Rightarrow \Delta x \Delta p = \frac{2\pi}{K} \cdot \hbar 2K = 4\pi\hbar \geq \frac{\hbar}{2}$

$\Rightarrow$ OK med Heisenberg!

III 26 $\quad$ a) $\phi(k) = \frac{N}{k^2 + \alpha^2}$ $\quad$ Koll: $\phi(k) \underset{k\to\infty}{\to} 0$, måste för vågfkt.

$$\Rightarrow \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk = \mathcal{F}^{-1}\left[\frac{N}{\alpha^2 + k^2}\right] = \left\{\begin{matrix}\text{BETA} \\ (\times\sqrt{2\pi})\end{matrix}\right\} =$$

$$= \frac{\sqrt{2\pi}\, N}{2\alpha} e^{-\alpha|x|} = \sqrt{\frac{\pi}{2}} \frac{N}{\alpha} e^{-\alpha|x|}$$

...

... b) Normalisering: $\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1$

dvs: $1 = \int_{-\infty}^{\infty} dx\, \frac{\pi}{2}\frac{N^2}{\alpha^2} e^{-2\alpha|x|} = \frac{\pi N^2}{\alpha^2}\int_0^{\infty} dx\, e^{-2\alpha x} =$

$= \frac{\pi N^2}{\alpha^2}\left(-\frac{1}{2\alpha}\right)\left(e^{-2\alpha\cdot\infty} - e^{-2\alpha\cdot 0}\right) = \frac{\pi N^2}{2\alpha^3}$

$\Leftrightarrow N = \sqrt{\frac{2\alpha^3}{\pi}} = \sqrt{\frac{2}{\pi}}\,\alpha^{3/2} \Rightarrow \psi(x) = \frac{\alpha^{3/2}}{\alpha} e^{-\alpha|x|} =$

$= \sqrt{\alpha}\, e^{-\alpha|x|}$

c) Heisenberg: $\Delta x \Delta p \geq \frac{\hbar}{2}$

$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ (dispersion) Vi har $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$.

Väntevärden: $\langle x\rangle = 0$ (ty $\psi(x)$ jämn)

$\langle x^2\rangle = \int_{-\infty}^{\infty}\psi^*(x)\, x^2 \psi(x)\, dx = \int_{-\infty}^{\infty} dx\, x^2 |\psi(x)|^2 =$

Ta bort belopp

$= \alpha\int_{-\infty}^{\infty} dx\, x^2 e^{-2\alpha|x|} = 2\alpha\int_0^{\infty} dx\, x^2 e^{-2\alpha|x|} = 2\alpha\frac{2}{(2\alpha)^3} = \frac{1}{2\alpha^2}$

$\Rightarrow \boxed{\Delta x = \frac{1}{\sqrt{2}\,\alpha}}$

$\phi(k) = \sqrt{\frac{2\alpha^3}{\pi}}\frac{1}{k^2+\alpha^2}$; $\langle k\rangle = 0$

$\langle k^2\rangle = \int_{-\infty}^{\infty} dk\, k^2 |\phi(k)|^2 = \frac{2\alpha^3}{\pi}\int_{-\infty}^{\infty} k^2 \frac{1}{(k^2+\alpha^2)^2} = \frac{2\alpha^3}{\pi}\cdot\frac{\pi}{2\alpha} = \alpha^2$

$\Rightarrow \Delta k = \alpha \Rightarrow \Delta p = \hbar\alpha \Rightarrow \Delta x \Delta p = \frac{\hbar}{\sqrt{2}} \geq \frac{\hbar}{2}$ OK!

III 30 Grupphastighet $v_g = \frac{d\omega}{dk} = \frac{d\omega}{d\nu}\frac{d\nu}{d\lambda}\frac{d\lambda}{dk}$

$\omega = 2\pi\nu \Rightarrow \frac{d\omega}{d\nu} = 2\pi$

$\lambda = \frac{c}{(\nu^2-\nu_0^2)^{1/2}} \Rightarrow \frac{d\nu}{d\lambda} = \frac{1}{d\lambda/d\nu} = \left(c\cdot 2\nu\cdot\left(-\frac{1}{2}\right)(\nu^2-\nu_0^2)^{-3/2}\right)^{-1} =$

$= -\frac{(\nu^2-\nu_0^2)^{3/2}}{c\nu}$

$k = \frac{2\pi}{\lambda} \Rightarrow \frac{d\lambda}{dk} = \frac{1}{dk/d\lambda} = \left(-\frac{2\pi}{\lambda^2}\right)^{-1} = -\frac{\lambda^2}{2\pi} = -\frac{1}{2\pi}\cdot\frac{c^2}{\nu^2-\nu_0^2}$

$\therefore v_g = 2\pi\frac{(\nu^2-\nu_0^2)^{3/2}}{c\nu}\frac{1}{2\pi}\frac{c^2}{(\nu^2-\nu_0^2)} = c\frac{(\nu^2-\nu_0^2)^{1/2}}{\nu} =$

$= c\sqrt{\frac{\nu^2-\nu_0^2}{\nu^2}} = c\sqrt{1-\frac{\nu_0^2}{\nu^2}} < c$

Men $v_f = \frac{\omega}{k} = \frac{2\pi\nu}{2\pi/\lambda} = \lambda\nu = \frac{c\nu}{\sqrt{\nu^2-\nu_0^2}} = c\sqrt{\frac{\nu^2}{\nu^2-\nu_0^2}} =$

$= \frac{c}{\sqrt{1-\frac{\nu_0^2}{\nu^2}}} > c$

OBS att $v_g v_f = c^2$