1. (10 points) Your colleague, Dr. Knowitall, has some quick-fire questions for you.

   (a) (1 point) What wavefunctions satisfy the *time-independent* Schrödinger equation?

   **Solution:** Only the wavefunctions of energy eigenstates (stationary states).

   (b) (1 point) What wavefunctions satisfy the *time-dependent* Schrödinger equation?

   **Solution:** All wavefunctions must satisfy the TDSE.

   (c) (1 point) If the quantum state of a system is given by an energy eigenstate at $t = 0$, how does it evolve in time?

   **Solution:** Via the “wiggle factor”: $\psi(t) = \psi(t = 0) \exp(-iEt/\hbar)$ where $E$ is the energy eigenvalue.

   (d) (2 points) What are the wavefunctions in the position and momentum representations, $\psi(x)$ and $\tilde{\psi}(p)$, of a free particle with definite momentum $p_0$?

   **Solution:** $\psi(x) = \exp(ip_0x/\hbar)/\sqrt{2\pi\hbar}$ and $\tilde{\psi}(p) = \delta(p - p_0)$.

   (e) (2 points) Give two important properties of Hermitian operators.

   **Solution:** They have real eigenvalues, eigenstates (kets/vectors) corresponding to different eigenvalues are orthogonal to each other, the set of eigenstates forms a complete basis.
(f) (2 points) Why are these properties necessary for Hermitian operators to represent physical observables?

**Solution:** The eigenvalues represent the possible outcomes of a measurement (which must therefore be real). Eigenstates with different eigenvalues are orthogonal to each other, so different physical outcomes do not overlap. A complete set means that any physical state within the Hilbert space of the system can be represented by a linear combination of the basis states.

(g) (1 point) The three position operators, \( \hat{x} \), \( \hat{y} \) and \( \hat{z} \), and the three momentum operators \( \hat{p}_x \), \( \hat{p}_y \) and \( \hat{p}_z \), can be used to form 15 distinct commutators. How many are zero?

**Solution:** The only non-zero ones are \([x, p_x] = [y, p_y] = [z, p_z] = i\hbar\), so 12.

2. (6 points) The quantum state of a spin-1 particle is given by

\[
|\psi\rangle = -\frac{i}{\sqrt{3}} |m = 0\rangle + \sqrt{\frac{2}{3}} |m = 1\rangle, \tag{1}
\]

(a) (2 points) If the component of the spin parallel to \( z \), \( \hat{S}_z \), is measured, what are the possible outcomes and the probabilities of those outcomes?

**Solution:** Possible outcomes are 0 and \( \hbar \), with probabilities 1/3 and 2/3 respectively.

(b) (4 points) If the component of the spin parallel to \( y \), \( \hat{S}_y \), is measured, what are the possible outcomes and the probabilities of those outcomes?

**Solution:** We need to find the eigenstates of \( \hat{S}_y \). Use the Pauli matrix representation and solve the eigenvalue equation

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{2}
\]

for \( \lambda = -1, 0 \) and 1. We obtain \((-1, i\sqrt{2}, 1)^T/2, (1, 0, 1)^T/\sqrt{2}, (-1, -i\sqrt{2}, 1)^T/2\), respectively. Now solve \(|\psi\rangle = a |m_y = -1\rangle + b |m_y = 0\rangle + c |m_y = 1\rangle\) to obtain \(a = 0\), \(b = \sqrt{1/3}\) and \(c = \sqrt{2/3}\). Therefore in a measurement of the spin component along \( y \), we obtain 0 with 1/3 probability and \( +\hbar \) with 2/3 probability.
3. (12 points) A particle of mass $m$ is trapped in a narrow, but very deep, potential well at $x = 0$. We will model this potential well as a Dirac $\delta$ function $V(x) = V_0 \delta(x)$.

(a) (3 points) The energy of the bound state is $E = -\frac{m V_0^2}{2 \hbar^2}$. Show that the wavefunction in the position representation $\psi(x) = N \exp(-\alpha x)$ for $x > 0$ and $N \exp(\alpha x)$ for $x < 0$, where $\alpha$ and $N$ are constants to be determined.

**Solution:** Plug the wavefunction into the TISE (working point):

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) = E \psi(x)$$

To obtain $\alpha = \frac{m V_0}{\hbar^2}$. The wavefunction must be normalised, $\int_{-\infty}^{\infty} \psi(x)^2 dx = 1$, so $N = \sqrt{\alpha} = \sqrt{m V_0 / \hbar^2}$.

(b) (5 points) Find the wavefunction in the momentum representation $\tilde{\psi}(p)$.

**Solution:** Fourier-transform the wavefunction in the position representation ($k = p/\hbar$):

$$\tilde{\psi}(p) = \langle p | \psi \rangle = \int_{-\infty}^{\infty} \langle x | p \rangle^* \langle x | \psi \rangle \, dx$$

$$= \sqrt{\frac{\alpha}{2\pi \hbar}} \left[ \int_{-\infty}^{0} e^{-ikx} e^{\alpha x} \, dx + \int_{0}^{\infty} e^{-ikx} e^{-\alpha x} \, dx \right]$$

$$= \sqrt{\frac{\alpha}{2\pi \hbar}} \left[ \frac{1}{\alpha - ik} + \frac{1}{\alpha + ik} \right]$$

$$= \sqrt{\frac{2}{\pi \hbar}} \frac{\alpha^{3/2}}{\alpha^2 + k^2}$$

(c) (2 points) Using your answer to part (b), find the expectation value of the squared momentum $\langle p^2 \rangle$. (You may use your answer to part (a) instead – but be very careful in your working.)

**Hint:** $\int_{-\infty}^{\infty} p^2 / [p^2 + b^2]^2 \, dp = \pi / (2b)$.

**Solution:** Working in momentum space, we have $\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 \tilde{\psi}(p)^2 \, dp$:

$$\langle p^2 \rangle = \frac{2\alpha^3 \hbar^2}{\pi} \int_{-\infty}^{\infty} \frac{k^2}{(\alpha^2 + k^2)^2} \, dk = \alpha^2 \hbar^2 = \frac{m^2 V_0^2}{\hbar^2}.$$

---

Page 3
(d) (2 points) Show that the bound state satisfies the uncertainty relation.

Hint: \( \int_{-\infty}^{\infty} x^2 \exp(-\kappa x) dx = 2/k^3 \).

**Solution:** The position uncertainty is \( \langle x^2 \rangle = 2\alpha \int_{0}^{\infty} x^2 e^{-2\alpha x} dx = 1/(2\alpha^2) \), using the symmetry \( \psi(-x) = \psi(x) \). We have therefore that \( \sigma_x^2 \sigma_p^2 = \langle x^2 \rangle \langle p^2 \rangle = \hbar^2 / 2 > \hbar^2 / 4 \).

4. (6 points) The wavefunction of the electron in a hydrogen atom, at \( t = 0 \), is given by

\[
\psi(r, \theta, \varphi) = \sqrt{\frac{15}{16\pi}} R(r) \sin^2 \theta \cos(2\varphi) \tag{8}
\]

where \( R(r) \) is a function of radius \( r \).

(a) (3 points) If the squared magnitude of the orbital angular momentum (\( L^2 \)) is measured at \( t = 0 \), what are the possible results and the probabilities to obtain those results?

**Solution:** \( |\psi\rangle = \frac{1}{\sqrt{3}}(|\ell = 2, m = +2\rangle + |\ell = 2, m = -2\rangle) \) so there is 100% probability to obtain \( L^2 = 2(2 + 1)\hbar^2 = 6\hbar^2 \).

(b) (3 points) If the \( z \)-component of the orbital angular momentum (\( L_z \)) is measured at \( t = 0 \), what are the possible results and the probabilities to obtain those results?

**Solution:** Using the above representation, we find there is a 50% probability to obtain \( L_z = -2\hbar \) or \( +2\hbar \).

5. (8 points) A particle of mass \( m \) is trapped in a 1D harmonic oscillator of natural frequency \( \omega \), which is perturbed by a weak potential \( V(x) = (m\omega/\hbar)V_0x^2 \).

(a) (2 points) What is the exact change to the energy of the \( n \)th level?

**Solution:** \( V(x) = \frac{1}{2}m\omega^2x^2 \rightarrow \frac{1}{2}m\omega^2x^2[1 + 2V_0/(\hbar\omega)] \). Thus the natural frequency shifts by \( \omega \rightarrow \omega\sqrt{1 + 2V_0/(\hbar\omega)} \) and the exact change in the energy level is

\[
\Delta E_n = \hbar\omega \left(n + \frac{1}{2}\right) \left[ \sqrt{1 + \frac{2V_0}{\hbar\omega}} - 1 \right]. \tag{9}
\]

(b) (4 points) Use first-order perturbation theory to determine the change in the energy of the \( n \)th level. Show that your result is consistent with your answer to part (a).

**Solution:** The first-order correction is \( \Delta E_n = \langle n | \hat{H}' | n \rangle \), where \( \hat{H}' = \frac{1}{2}V_0(\hat{a} + \hat{a}^\dagger)^2 \). Thus we obtain \( \Delta E_n = V_0(n + 1/2) \). This is consistent with expanding \( \sqrt{1 + 2V_0/(\hbar\omega)} \approx V_0/(\hbar\omega) \).
(c) (2 points) What condition must $V_0$ satisfy for perturbation theory to be accurate? Explain your reasoning.

**Solution:** Either: PT works while the energy shift is small compared to the energy itself, i.e. $\Delta E_n / E_n \approx V_0 / (\hbar \omega) \ll 1$. Or: expanding to second order, $\Delta E_n = V_0(n + 1/2) - (n + 1/2)V_0^2 / (2\hbar \omega)$, the next-order correction is small if $V_0 / (2\hbar \omega) \ll 1$.

6. (a) (2 points) What physical transformation is associated with the parity operator $\hat{\Pi}$? What does this operator do, when applied to the state (or wavefunction) describing a quantum-mechanical system?

**Solution:** The parity operator inverts the spatial axes $x \rightarrow -x$, etc. In terms of position eigenstates, $\hat{\Pi} |x\rangle = |-x\rangle$, or in the position representation, $\langle x | \hat{\Pi} | \psi \rangle = \hat{\Pi} \psi (x) = \psi (-x)$.

(b) (3 points) By considering how the expectation value of position, $\langle x \rangle$, changes under parity transformation (or otherwise), show that $[\hat{\Pi}, \hat{x}] = 2\hat{\Pi}\hat{x}$.

**Solution:** The expectation value of the position changes as $\langle \psi' | x | \psi \rangle = -\langle \psi | x | \psi \rangle$ if $|\psi'\rangle = \Pi |\psi\rangle$. Therefore $\hat{\Pi}^\dagger \hat{x} \hat{\Pi} = -\hat{x}$. The parity operator is unitary, so apply $\hat{\Pi}$ to both sides, obtaining $\hat{x} \hat{\Pi} = -\hat{\Pi} \hat{x}$. QED. Alternatively, in the position representation, $\langle x | [\hat{\Pi}, \hat{x}] | \psi \rangle = \hat{\Pi} [x \psi (x)] - \hat{x} [\psi (-x)] = -2x \psi (-x) = \langle x | 2\hat{\Pi}\hat{x} | \psi \rangle$.

(c) (3 points) Under what conditions does the parity operator commute with the kinetic and potential energy operators $\hat{T}$ and $V(\hat{x})$? What does this mean for the wavefunctions $\psi (x) = \langle x | E \rangle$ of the energy eigenstates $|E\rangle$?

**Solution:** The parity operator always commutes with the kinetic energy operator. The parity operator commutes with the potential energy operator iff the potential $V(x)$ even, i.e. $V(x) = V(-x)$. If $[\hat{\Pi}, \hat{H}] = 0$, it is possible to construct eigenstates that are simultaneously eigenstates of parity and the Hamiltonian; the wavefunctions of these eigenstates would be either even or odd.

END
Formulas

- The Dirac delta function:
  \[ f(a) = \int_{-\infty}^{\infty} \delta(x - a) f(x) \, dx, \quad \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dx \]  
  (10)

- Creation and annihilation operators for the harmonic oscillator, \( V(\hat{x}) = \frac{1}{2} m \omega^2 \hat{x}^2 \):
  \[ \hat{a}^\dagger = \frac{\hat{x}}{2L} - \frac{iL\hat{p}}{\hbar}, \quad \hat{a} = \frac{\hat{x}}{2L} + \frac{iL\hat{p}}{\hbar} \]  
  (11)

  where \( L = \sqrt{\hbar/(2m\omega)} \).

- Pauli matrices for \( j \) or \( s = 1/2 \) (\( \hat{J}_i | \hat{S}_i = \frac{\hbar}{2} \sigma_i \)):
  \[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  
  (12)

- Pauli matrices for \( j, \ell \) or \( s = 1 \) (\( \hat{J}_i | \hat{L}_i | \hat{S}_i = \hbar \sigma_i \)):
  \[ \sigma_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]  
  (13)

- The Hamiltonian in spherical polar coordinates:
  \[ \hat{H} = \frac{\hat{p}_r^2}{2m} + \frac{\hat{L}^2}{2mr^2} + V(r), \]  
  (14)

  where
  \[ \hat{p}_r = -\frac{\hbar}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \]  
  (15)

  \[ \hat{L}^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \]  
  (16)