TIF 150: Information theory for complex systems

<u>Time</u>: March 14, 2014, 14.00-18.00
<u>Allowed material</u>: Calculator (type approved accordingly to Chalmers rules).
<u>Teacher (available during exam)</u>: Kristian Lindgren (7723131)
<u>Examiner</u>: Kristian Lindgren

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, 42p for 5. Points from homework problems and project may be included, but a minimum of 20p is required on the written exam.
The results will be available on April 4.

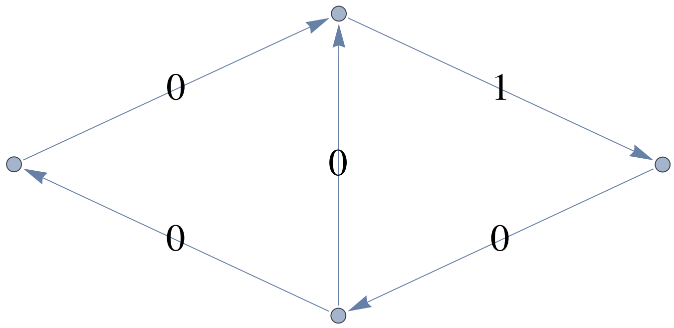
1. Finding the deviating ball 8p
You have 13 almost identical balls. The only difference between them is that one of them have a slightly deviating weight, it could be heavier, or lighter. You are tasked to find which of the balls that deviates, with only the means of a balance scale (that tips to the side with the heaviest load, or remains in balance if the two sides are of equal weight) and three measurements.

a) How large is the entropy of the system? How large is the entropy of which of the 13 balls that deviates?
b) Assuming ideal measurements, how many measurements would you need at least to find the deviating one?
c) Find a procedure to identify the deviating ball using a balance scale and only three measurements. You need only to describe the procedure assuming the worst case outcome for each measurement (the most probable outcome), i.e. not all possible branches of measurements.

2. Rubber band. 10p
A well known elasticity model for rubber bands is a one dimensional system of cells/parts that can either be contracted or extended. The contracted elements can be called C and the extended ones can be called E. Let the **extended** ones contribute with -J to the total energy of the system and have length a, the **contracted** ones have length $a/2$ and contribute with 0 to the total energy. If the average length is L, what is the equilibrium distribution (You may give the answer as a function of temperature)? What happens if you heat a rubberband?

3. CA information. 8p

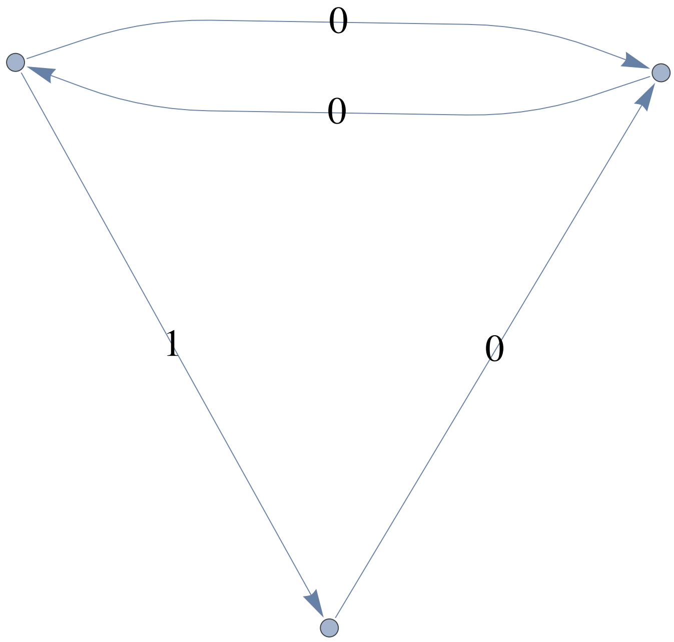
Consider a one-dimensional cellular automaton given by rule 238 (100 and 000 maps to 0, the rest to 1). Let the initial state be characterized by the following finite state automaton (where it is assumed that if two arcs leave the same node, they have equal probabilities).



a) What is the initial entropy (at t=0)?
b) Derive the finite state automaton that characterizes the CA state after one time step (t=1).
c) What is the entropy at (t=1)? At (t=infinity)?

4. Correlation complexity. 12p

Below are a hidden Markov model, when two arcs leave a node it is assumed that they have the same probability.



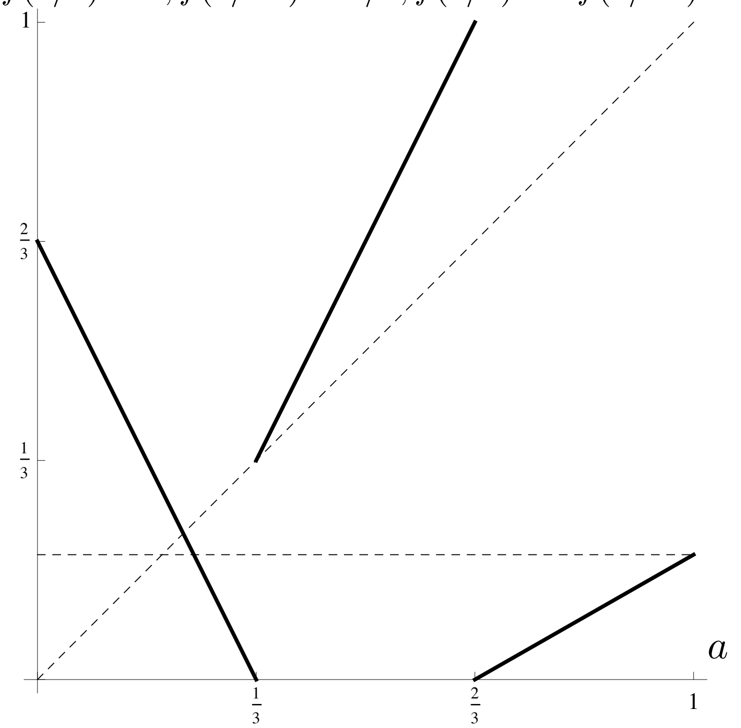
a) How long correlations (in information theoretical terms) are present in the system?

b) Determine the correlation complexity.

5. **Chaos and information. 12p**

Let a piecewise linear map be defined by the figure below, where a <1 and where the mapping is determined by:

$$f(0) = 2/3, f(1/3) = 0, f(1/3^+) = 1/3, f(2/3) = 1f(2/3^+) = 0, f(1) = a$$



Consider the dynamic system given by

$$x_{t+1} = f(x_t)$$

a) Start with a close to 0 and let a increase. Determine whether there is a stable fix point, stable periodic orbit (you don't have to find the periodicity) or chaos. At what value for a is there a change to the dynamical characteristics?
b) Suppose that a = 1/3. Determine the invariant measure that characterizes the chaotic behavior, and calculate the Luyapunov exponent. Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

Information theory for complex systems – useful equations

Basic quantities

$$I(p) = \log\frac{1}{p} \qquad S[P] = \sum_{i=1}^{n} p_i \log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right] = \sum_{i=1}^{n} p_i \log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k+1$ constraints) using the Lagrangian L

$$L(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu) = S[P] + \sum_{k=1}^{r} \lambda_k \left(F_k - \sum_{i=1}^{n} p_i f_k(i)\right) + (\mu-1)\left(1 - \sum_{i=1}^{n} p_i\right) ,$$

$$p_j = \exp\left(-\mu - \sum_{k=1}^{r} \lambda_k f_k(j)\right), \quad \mu(\lambda) = \ln\sum_{j=1}^{n} \exp(-\sum_{k} \lambda_k f_k(j)), \quad \frac{\partial\mu(\lambda)}{\partial\lambda_k} = -F_k$$

Symbol sequences

$$s = \lim_{n\to\infty} \sum_{x_1\ldots x_{n-1}} p(x_1\ldots x_{n-1}) \sum_{x_n} p(x_n \mid x_1\ldots x_{n-1}) \log\frac{1}{p(x_n \mid x_1\ldots x_{n-1})} =$$

$$= \lim_{n\to\infty}(S_n - S_{n-1}) = \Delta S_\infty = \lim_{n\to\infty}\frac{1}{n} S_n$$

$$k_1 = \log\nu - S_1, \qquad k_n = -S_n + 2S_{n-1} - S_{n-2} = -\Delta S_n + \Delta S_{n-1} = -\Delta^2 S_n \quad (n = 2,3,\ldots)$$

$$S_{\mathrm{tot}} = \log\nu = \sum_{m=1}^{\infty} k_m + s$$

$$\eta = \sum_{m=1}^{\infty} (m-1)k_m = \lim_{m\to\infty}\lim_{n\to\infty} \sum_{\sigma_m} p(\sigma_m) \sum_{\tau_n} p(\tau_n \mid \sigma_m) \log\frac{p(\tau_n \mid \sigma_m)}{p(\tau_n)} = \lim_{m\to\infty}(S_m - m\, s)$$

Geometric information theory

$$p(r;x) = \frac{1}{\sqrt{2\pi}\, r}\int_{-\infty}^{\infty} dw\, e^{-w^2/2r^2} p(x-w) \quad , \qquad \left(-r\frac{\partial}{\partial r} + r^2\frac{d^2}{dx^2}\right) p(r;x) = 0$$

$$p_{\mathrm{Gaussian}}(r;x) = \frac{1}{\sqrt{2\pi}\sqrt{b^2+r^2}} \exp\left(-\frac{x^2}{2(b^2+r^2)}\right)$$

$$K[p_0;p] = \int dx\, p(x) \ln\frac{p(x)}{p_0(x)} = \int_0^{\infty} \frac{dr}{r} \int dx\, k(r,x), \qquad k(r,x) = r^2 p(r;x) \left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r) = D_{\mathrm{E}} - r\frac{\partial}{\partial r}\int d\mathbf{x}\, p(r;\mathbf{x}) \ln\frac{1}{p(r;\mathbf{x})}$$

Chemical systems and information flow

$$E = k_{\mathrm{B}} T_0 \frac{N}{V} K, \qquad K = \int_V d\mathbf{x}\, K[c_0;c(\mathbf{x})] = \int_V d\mathbf{x} \sum_{i=1}^{M} c_i(\mathbf{x}) \ln\frac{c_i(\mathbf{x})}{c_{i0}}, \quad \sum_{i=1}^{M} c_i(\mathbf{x},t) = 1$$

$$K = V\sum_{i=1}^{M}\bar{c}_i \ln\frac{\bar{c}_i}{c_{i0}} + \int_V d\mathbf{x}\sum_{i=1}^{M} c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{\bar{c}_i} = K_{\text{chem}} + K_{\text{spatial}}$$

$$K_{\text{spatial}} = \int_0^\infty \frac{\partial r}{r}\int d\mathbf{x}\, k(r,\mathbf{x})$$

$$k(r,\mathbf{x}) = r^2\sum_i \tilde{c}_i(r,\mathbf{x})\left(\nabla\ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}\right)^2 = r^2\sum_i\frac{(\nabla\tilde{c}_i(r,\mathbf{x}))^2}{\tilde{c}_i(r,\mathbf{x})} = \left(-r\frac{\partial}{\partial r} + r^2\nabla^2\right)\sum_i \tilde{c}_i(r,\mathbf{x})\ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}$$

$$\dot{c}_i(\mathbf{x},t) = \frac{d}{dt}c_i(\mathbf{x},t) = D_i\nabla^2 c_i(\mathbf{x},t) + F_i(\mathbf{c}(\mathbf{x},t)) + b_i(c_{i,\text{ res}} - c_i(\mathbf{x},t))$$

$$\sigma(\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla c_i(\mathbf{x},t))^2}{c_i(\mathbf{x},t)} - F_i(\mathbf{c}(\mathbf{x},t))\ln\frac{c_i(\mathbf{x},t)}{c_{i0}}\right)$$

$$j_r(r,\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla\tilde{c}_i(r,\mathbf{x},t))^2}{\tilde{c}_i(r,\mathbf{x},t)} - \tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}}\right),\ \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) = \exp(\tfrac{r^2}{2}\nabla^2)F_i(\mathbf{c}(\mathbf{x},t))$$

$$\mathbf{j}(r,\mathbf{x},t) = -r^2\nabla\sum_i\tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}},\quad J(r,x,t) = -\sum_i b_i(\tilde{c}_i + c_{i,\text{ res}})[r\nabla\ln\tilde{c}_i]^2$$

$$\dot{k}(r,\mathbf{x},t) = r\frac{\partial}{\partial r}j_r(r,\mathbf{x},t) - \nabla\cdot\mathbf{j}(r,\mathbf{x},t) + J(r,x,t)$$

Chaos and information

$$\int dx\mu(x)\varphi(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}\varphi(f^k(x(0))),$$

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}\ln|f'(x(k))| = \int dx\mu(x)\ln|f'(x)|$$

$$h(\mu,\mathbf{A}) = \lim_{n\to\infty}\frac{1}{n}H(B^{(n)}) = \lim_{n\to\infty}\left(H(B^{(n+1)}) - H(B^{(n)})\right),\quad s_\mu = \lim_{diam(\mathbf{A})\to 0}h(\mu,\mathbf{A}),\ \ s_\mu = \lambda$$

2014-① 13 balls, one deviates: heavy or light

entropy of system $S = \log(2 \cdot 13) = \log 26$
entropy of problem (finding the deviating one): $S = \log 13$

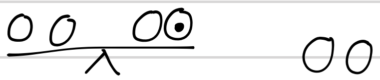
ideal measurments are arranged so that you get $\log 3$ information, i.e., equally probable outcomes!

Then, theoretically, 3 measurments may work since $3 \cdot \log 3 > \log 13$.

4 vs 4, 5 aside → "worst" or most probable outcome is "balanced"

5 remains $S_1 = \log 5 < 2 \log 3$, OK.

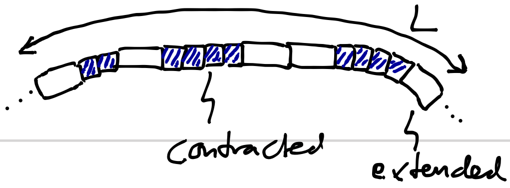
take three of these and a normal one ⊙



balanced result most probable: $S_2 = \log 2 < \log 3$ ok!

on of the two against ⊙ determines the deviating one. $\underline{S_3 = 0}$

2014-2 Rubber band



$h(C) = -J$, $h(E) = 0$

$\underbrace{}_{\text{contracted}}$ length $a/2$; $\underbrace{}_{\text{extended}}$ length a

contracted → length $a/2$

extended → length a

You can pick either an energy constraint <u>or</u> a length constraint (since the length constraint <u>is</u> an energy constraint).

with energy $u = -J\,p(E)$ and with

No interactions between neighbour states you get:

$$p(E) = \frac{e^{\beta J}}{1+e^{\beta J}} \quad \text{and} \quad p(C) = \frac{1}{1+e^{\beta J}}$$

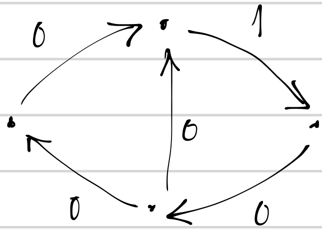
where $\beta = \frac{1}{k_B T}$ is Lagrange variable for the energy constraint

$$\text{average length } L = p(C)\frac{a}{2} + p(D)\,a = \frac{\frac{1}{2} + e^{\beta J}}{1+e^{\beta J}} = 1 - \frac{1}{2}\,\frac{1}{1+e^{\beta J}}$$

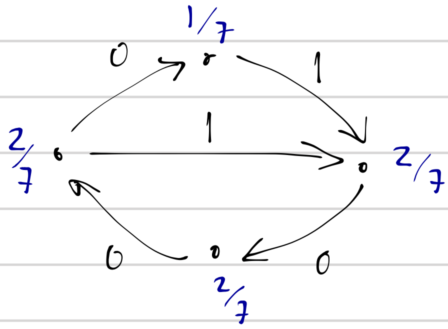
So if T increases, β decreases, and <u>L decreases</u>!

2014-3 CA-rule 238: $000 \to 0$, $100 \to 0$, oth $\to 1$

$t=0$



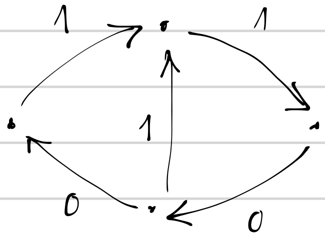
equivalent to



infinitely long preceding sequence determines (almost always) which node we are in an entropy then given by the entropy of choices on the nodes $s = \frac{2}{7} \log 2$ (left node)

CA rule

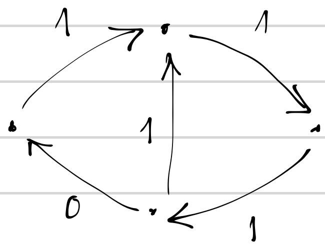
$t=1$



As before, we find $s = \frac{2}{7} \log 2$.

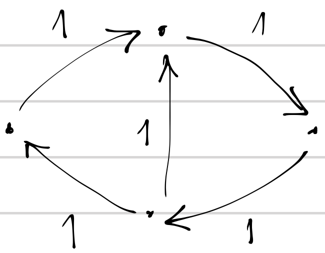
CA rule

$t=2$



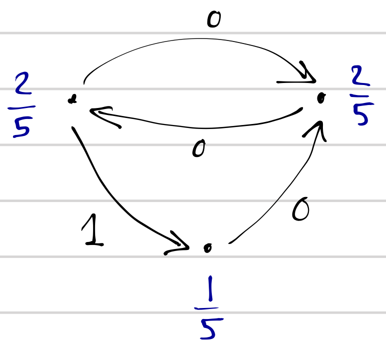
CA rule

$t \geq 3$



$\Longrightarrow$ $s = 0$

2014-④ Correlation complexity



$$k_m = \sum p(x_1 \ldots x_{m-1}) \sum_{x_m} p(x_m|x_1..x_{m-1}) \log \frac{p(x_m|x_1..x_{m-1})}{p(x_m|x_2..x_{m-1})}$$

$k_m = 0$ only if $p(x_m|x_1..x_{m-1}) =$
$= p(x_m|x_2..x_{m-1})$
for all sequences

But this equality cannot hold when $x_2 \ldots x_{m-1}$ are only 0's, since $x_1 = 1$ will determine which node we are in. $\rightarrow$ $k_m > 0$ all m.

Repeating the derivation in the lecture notes we find:

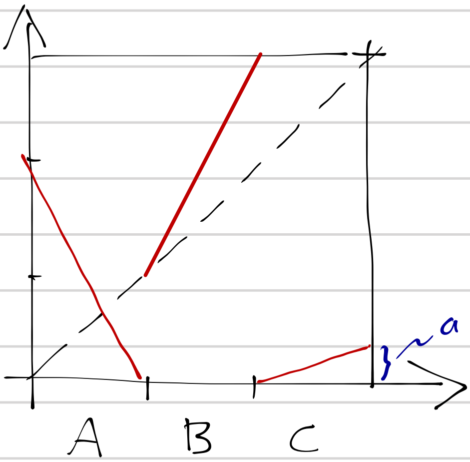
$$\eta = \sum_z p(z)\log\frac{1}{p(z)} - \lim_{m\to\infty} \sum_{T_m} p(T_m) \sum_z p(z|T_m)\log\frac{1}{p(z|T_m)}$$

where z is a node and T_m is the following sequence. (In the derivation we have used the fact that a preceding sequence σ_m almost always determines the node z in the limit $m \to \infty$.)

The second expression quantifies the uncertainty of node z when we observe T (in average). If T starts w. an even number of 0's we cannot know whether we started in the left or in the bottom node. The probability for such a sequence from the left node is $p(\text{left}) \cdot \frac{1}{2} = \frac{1}{5}$ and from the bottom node $p(\text{bottom}) = \frac{1}{5}$. This gives that the second expression is $\frac{2}{5}\log 2$ (since the uncertainty of z-node is one bit in this case. And

$$S = \frac{1}{5}\log 5 + 2\cdot\frac{2}{5}\log\frac{5}{2} - \frac{2}{5}\log 2 = \log 5 - \frac{6}{5}\log 2$$

2014 - ⑤ Chaos Solution sketch!



$f'(A) = -2, \quad f'(B) = 2, \quad f'(C) = 3a$

small a : $A \to B \to C \to A \ldots$ $\qquad \mu(A) = \mu(B) = \mu(C) = \frac{1}{3}$

$$\lambda = \frac{1}{3}\left(\ln 2 + \ln 2 + \ln(3a)\right)$$

$\lambda < 0$ if $2 \cdot 2 \cdot 3a < 1 \Rightarrow a < \frac{1}{12}$

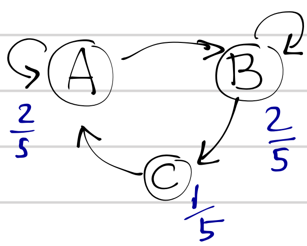
$\Rightarrow$ stable periodic orbit if $a < \frac{1}{12}$.

We also check that we stay within $A \to B \to C \to A \ldots$

$$x_0 = 1 \to \underbrace{x_1 = a}_{\in A} \to \underbrace{x_2 = \frac{2}{3} - 2a}_{\in B} \to x_3 = -\frac{1}{3} + 2\left(\frac{2}{3} - 2a\right) = \underbrace{1 - 4a}_{\in C}$$

We note that as $a > 1/12$, x_3 does not always get back in C, i.e., we spend more time in A and B which amplifies the value of λ, clearly showing that $\lambda > 0$ for $a > \frac{1}{12} \Rightarrow$ chaos.

For $a = \frac{1}{3}$, $\{A, B, C\}$ is a generating partition.



Invariant measure $\mu(A) = \mu(B) = \frac{2}{5}, \mu(C) = \frac{1}{5}$

$$\lambda = \frac{4}{5}\ln 2 + \frac{1}{5}\ln 1 = \frac{4}{5}\ln 2$$

$$S_\mu = 2 \cdot \frac{2}{5}\ln 2 = \frac{4}{5}\ln 2$$