TIF 150: Information theory for complex systems

Time: March 11, 2013, 8.30-12.30
Allowed material: Calculator (type approved accordingly to Chalmers rules).
Teacher (available during exam): Oskar Lindgren (0733-882112)
Examiner: Kristian Lindgren

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, 42p for 5. Points from homework problems and project may be included, but a minimum of 20p is required on the written exam.
The results will be available on March 26.

1. Ball measurements. 12p

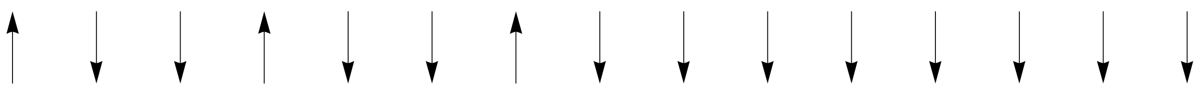
Assume that you have a production line producing balls at a fast pace. There is a probability of a small defect or contamination that renders the ball useless. The probability of this defect is 1/3 per ball (assume no correlation between balls). You have a measuring device in which you can put any number of balls and the device tells you if there is defective balls among them (a yes/no answer). The measuring process is a bottleneck and you want to find a strategy to reduce the number of measurements while still finding all defective balls.

a) How much is the entropy per produced ball? If you could arrange ideal measurements, how many measurements per ball (on average) would you need to find all deviating ones?
b) Using an information-theoretic approach, find a procedure distinguishing all deviating balls that is more effective than measuring every single ball one at a time.
c) How many measurements would you need per ball using this improved procedure?

(answers don't have to be in decimal form)

2. An equilibrium system. 8p

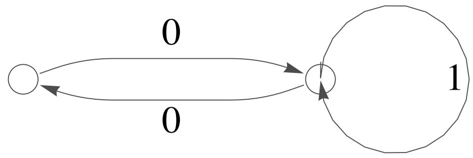
Consider a one dimensional spin system where each spin can be either Up or Down.



There is a local contribution to the energy, being $-J<0$ when a "Up" particle is followed by a "Down" particle (if the system is read left to right), no other pairs contribute with energy. If the average energy is u, what is the equilibrium distribution? Or, in other words, what are the probabilities over sequences of spins that characterize the system? (You may give the answer as a function of temperature instead of energy.) What are the probabilities in the limit of zero temperature?

3. CA information. 10p

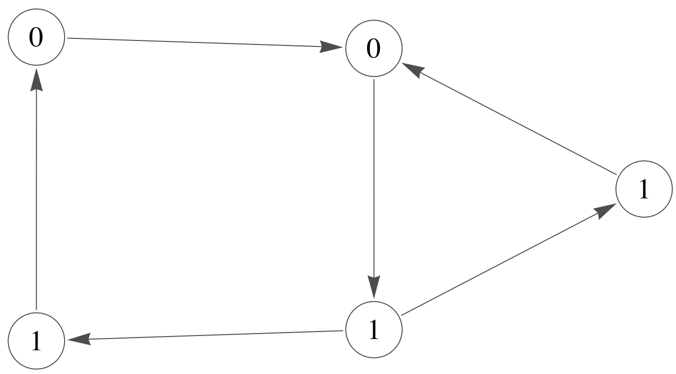
Consider a one-dimensional cellular automaton given by rule 150 (i.e. neighborhoods 111, 001, 010, 100 map to 1). Let the initial state be characterized by the following finite state automaton (where it is assumed that if two arcs leave the same node, they have equal probabilities).



a) What is the initial entropy (at t=0)?
b) Derive the finite state automaton that characterizes the CA state after one time step (t=1).
c) What is the entropy at (t=1) and at (t=2).

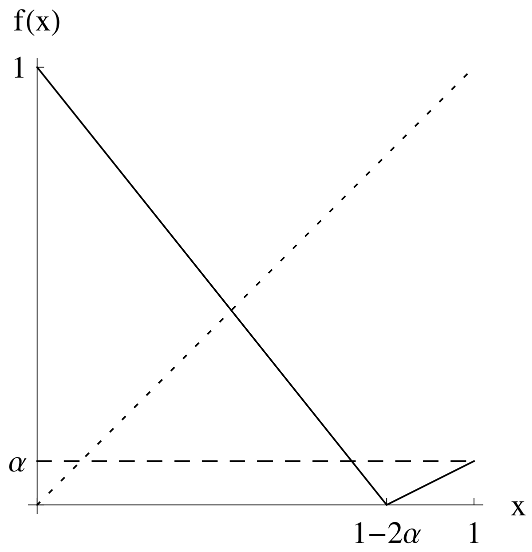
4. Correlation complexity. 10p

Below are a hidden Markov model, when two arcs leave a node it is assumed that they have the same probability. Determine the correlation complexity .



5. **Chaos and information. 10p**

Let a piecewise linear map be defined by the figure below, where alpha <0.5 and where the mapping is determined by $f(0) = 1, f(1 - 2\alpha) = 0, f(1) = \alpha$



Consider the dynamic system given by

$$x_{t+1} = f(x_t)$$

a) Start with *alpha* close to 0 and let *alpha* increase. Determine whether there is a stable fix point, stable periodic orbit (you don't have to consider orbits alternating between three or more points), or chaos. What are the critical values for *alpha* (for which there is a change in dynamical characteristics)?
b) Suppose that *alpha* = 1/3. Determine the invariant measure that characterizes the chaotic behavior, and calculate the Luyapunov exponent. Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

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② Spin system

config.	energy	notation
↑↓	$-J$	p_0
↓↑	0	p_1
↑↑	0	p_2
↓↓	0	p_3

$$\begin{cases} p(\uparrow) = p_0 + p_2 = p_1 + p_2 \Rightarrow p_0 = p_1 \\ p(\downarrow) = p_0 + p_3 \end{cases}$$

$$s = \Delta S_2 = 2p_0 \ln\frac{1}{p_0} + p_2 \ln\frac{1}{p_2} + p_3 \ln\frac{1}{p_3} - (p_0+p_2)\ln\frac{1}{p_0+p_2} - (p_0+p_3)\ln\frac{1}{p_0+p_3}$$

constraints energy: $-p_0 J = u \quad (\beta)$

norm. $2p_0 + p_2 + p_3 = 1 \quad (\mu)$

$$L = s + \beta(u + p_0 J) + \mu(1 - 2p_0 - p_2 - p_3)$$

$$\frac{\partial L}{\partial p_0} = 0 \Rightarrow -2 - 2\ln p_0 + 1 + \ln(p_0+p_2) + 1 + \ln(p_0+p_3) + \beta J - 2\mu = 0$$

$$\Rightarrow \frac{p_0^2}{(p_0+p_2)(p_0+p_3)} = e^{-2\mu+\beta J} \qquad (*)$$

$$\frac{\partial L}{\partial p_2} = 0 \Rightarrow \frac{p_2}{p_0+p_2} = e^{-\mu} \quad (**) \qquad \text{and} \qquad \frac{\partial L}{\partial p_3} = 0 \Rightarrow \frac{p_3}{p_0+p_3} = e^{-\mu} \quad (***)$$

$\Rightarrow p_2 = p_3$

Take eq(*)/eq(**)/eq(***): $\dfrac{p_0^2}{p_2^2} = e^{\beta J}$

but $p_2 = \frac{1}{2}(1-2p_0)$: $\dfrac{p_0}{\frac{1}{2}(1-2p_0)} = e^{\beta J/2}$

$$\Rightarrow p_0 = \frac{1}{2}\frac{e^{\beta J/2}}{1+e^{\beta J/2}}$$

etc ....

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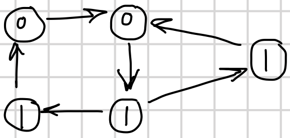
ALTERNATIVE SOLUTION

(4)



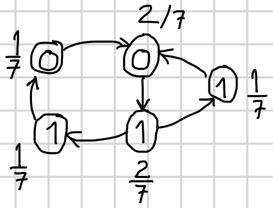
It turns out that correlation length only goes to $m=3$ here.

This suggests we can calculate η by

$$\eta = \sum_{m=2}^{\infty}(m-1)k_m = k_2 + 2k_3$$

(which should be simpler in this situation, compared to the solution I did in the lecture!)

We need:
- distribution over nodes
- S_1 based on P_1
- $k_1 = \log 2 - S_1$
- S_2 based on P_2
- $k_2 = -S_2 + 2S_1$
- entropy s
- k_3 most simply found through $\log 2 = s + k_1 + k_2 + k_3$



$\Rightarrow$ P_1: $p(0) = \frac{3}{7}$, $p(1) = \frac{4}{7}$

P_2: $p(00) = \frac{1}{7}$, $p(01) = \frac{2}{7}$, $p(10) = \frac{2}{7}$, $p(11) = \frac{2}{7}$

$$S_1 = \log 7 - \frac{3}{7}\log 3 - \frac{8}{7}\log 2$$

$$k_1 = \log 2 - S_1 = \frac{3}{7}\log 3 + \frac{15}{7}\log 2 - \log 7$$

$$S_2 = \log 7 - \frac{6}{7}\log 2$$

$$k_2 = -S_2 + 2S_1 = \ldots = \log 7 - \frac{10}{7}\log 2 - \frac{6}{7}\log 3$$

$$k_3 = \ldots = \frac{3}{7}\log 3$$

and

$$\eta = k_2 + 2k_3 = \log 7 - \frac{10}{7}\log 2$$

(which is what we got with the other approach in the lecture)