Information theory for complex systems

Time: March 18, 2011, 14.00 – 18.00.
Allowed material: Calculator (type approved according to Chalmers rules).
Teacher (available during exam): Oskar Lindgren (0768638283)
Examiner: Kristian Lindgren (772 3131)

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, and 42p for 5. (Points from homework problems and project may be included, but a minimum of 20p is required on the written exam.)
The results will be available on April 1.

1. Balance measurements.
Assume you have 4 balls of different weight: $1+2\varepsilon$, $1+\varepsilon$, $1-\varepsilon$, $1-2\varepsilon$, where $\varepsilon = 0.01$. You have at your disposal 4 balance measurements to identify the balls. Find a procedure by using an information-theoretic approach. Answer also the following questions:

(a) What is the initial uncertainty (entropy)? Compare this with how much information you could gain from four ideal measurements.
(b) Explain why a procedure with 3 measurements does not work in general.

(10 p)

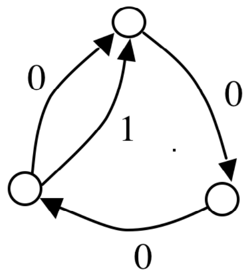
2. An equilibrium spin system. Consider an infinite one-dimensional lattice system where each site can be in one of 4 states (A, B, C, D).

...	A	A	B	A	C	D	A	D	C	B	B	D	A	...

There is a local contribution to the energy, being $-J < 0$ when two neighbouring states are the pairs (AB, BA, BC, CB, CD, DC, DA, AD), $+J$ when the states are (AC, CA, BD, DB). There is no interaction energy when the neighbouring states are the same. If the average energy is u, what is the equilibrium distribution? Or, in other words, what are the probabilities over sequences of symbols that characterise the system? (You may give the answer as a function of temperature instead of energy.) What are the probabilities in the limit of zero temperature?

(10 p)

3. **CA information.**
Consider a one-dimensional cellular automaton given by elementary rule R82, i.e., neighbourhoods 110, 100 and 001 map to 1, but all others to 0. Let the initial state be characterized by the following finite state automaton (where it is assumed that if two arcs leave the same node they have equal probabilities)

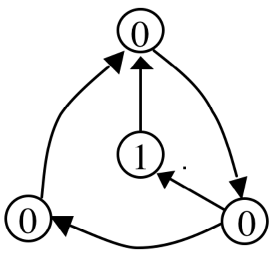


(a) What is the initial entropy (at $t = 0$)? Derive the finite state automaton that characterizes the CA state after one time step ($t = 1$). What is the entropy at this time?

(b) What is the entropy at $t = 2$?

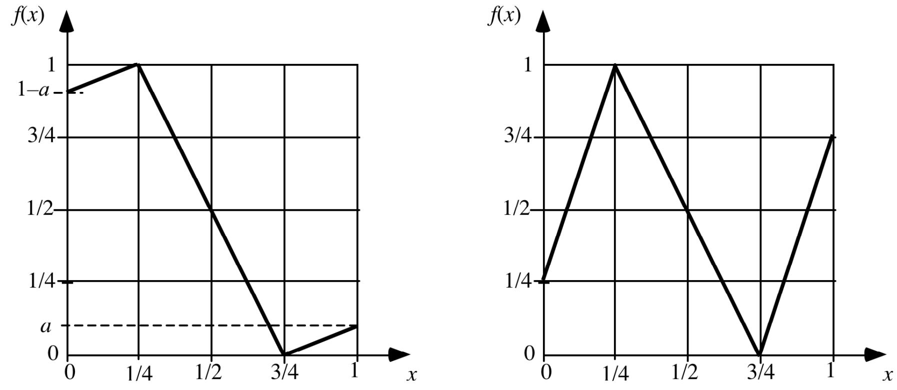
(10 p)

4. **Correlation complexity.** Determine the correlation complexity η for the process described by the hidden Markov model below. (When two arcs leave a node it is assumed that they have the same probability.)



(8 p)

5. Chaos and information. Let a piecewise linear map $f(x)$ be defined by the figure below, where $0 < a < 3/4$, and where the mapping is determined by $f(0) = 1 - a$, $f(1/4) = 1$, $f(3/4) = 0$, and $f(1) = a$.



(i) (ii)

Consider the dynamical system

$$x_{t+1} = f(x_t) .$$

(i) Start with a close to 0 and investigate how the dynamics changes when a is increased. Determine whether there is a stable fixed point, stable periodic orbit, or chaos. Are there critical values for a (for which there is a change in dynamical characteristics)?

(ii) Suppose now that $a = 3/4$. Determine the invariant measure that characterizes the chaotic behaviour, and calculate the Lyapunov exponent λ. Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

(12 p)

Information theory for complex systems – useful equations

Basic quantities

$$I(p)=\log\frac{1}{p} \qquad S[P]=\sum_{i=1}^{n}p_i\log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right]=\sum_{i=1}^{n}p_i\log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k+1$ constraints) using the Lagrangian L

$$L\left(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu\right)=S[P]+\sum_{k=1}^{r}\lambda_k\left(F_k-\sum_{i=1}^{n}p_i f_k(i)\right)+\left(\mu-1\right)\left(1-\sum_{i=1}^{n}p_i\right),$$

$$p_j=\exp\left(-\mu-\sum_{k=1}^{r}\lambda_k f_k(j)\right),\quad \mu(\lambda)=\ln\sum_{j=1}^{n}\exp(-\sum_{k}\lambda_k f_k(j)),\quad \frac{\partial\mu(\lambda)}{\partial\lambda_k}=-F_k$$

Symbol sequences

$$s=\lim_{n\to\infty}\sum_{x_1\ldots x_{n-1}}p(x_1\ldots x_{n-1})\sum_{x_n}p(x_n\mid x_1\ldots x_{n-1})\log\frac{1}{p(x_n\mid x_1\ldots x_{n-1})}=$$

$$=\lim_{n\to\infty}\left(S_n-S_{n-1}\right)=\Delta S_\infty=\lim_{n\to\infty}\frac{1}{n}S_n$$

$$k_1=\log\nu-S_1,\qquad k_n=-S_n+2S_{n-1}-S_{n-2}=-\Delta S_n+\Delta S_{n-1}=-\Delta^2 S_n\quad (n=2,3,\ldots)$$

$$S_{\text{tot}}=\log\nu=\sum_{m=1}^{\infty}k_m+s$$

$$\eta=\sum_{m=1}^{\infty}(m-1)k_m=\lim_{m\to\infty}\lim_{n\to\infty}\sum_{\sigma_m}p(\sigma_m)\sum_{\tau_n}p(\tau_n\mid\sigma_m)\log\frac{p(\tau_n\mid\sigma_m)}{p(\tau_n)}=\lim_{m\to\infty}(S_m-m\,s)$$

Geometric information theory

$$p(r;x)=\frac{1}{\sqrt{2\pi}\,r}\int_{-\infty}^{\infty}dw\,e^{-w^2/2r^2}p(x-w),\qquad \left(-r\frac{\partial}{\partial r}+r^2\frac{d^2}{dx^2}\right)p(r;x)=0$$

$$p_{\text{Gaussian}}(r;x)=\frac{1}{\sqrt{2\pi}\sqrt{b^2+r^2}}\exp\left(-\frac{x^2}{2(b^2+r^2)}\right)$$

$$K[p_0;p]=\int dx\,p(x)\ln\frac{p(x)}{p_0(x)}=\int_0^{\infty}\frac{dr}{r}\int dx\,k(r,x),\qquad k(r,x)=r^2p(r;x)\left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r)=D_{\text{E}}-r\frac{\partial}{\partial r}\int d\mathbf{x}\,p(r;\mathbf{x})\ln\frac{1}{p(r;\mathbf{x})}$$

Chemical systems and information flow

$$E=k_{\text{B}}T_0\frac{N}{V}K,\qquad K=\int_V d\mathbf{x}\,K[c_0;c(\mathbf{x})]=\int_V d\mathbf{x}\sum_{i=1}^{M}c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{c_{i0}},\quad \sum_{i=1}^{M}c_i(\mathbf{x},t)=1$$

$$K = V\sum_{i=1}^{M} \bar{c}_i \ln\frac{\bar{c}_i}{c_{i0}} + \int_V d\mathbf{x} \sum_{i=1}^{M} c_i(\mathbf{x}) \ln\frac{c_i(\mathbf{x})}{\bar{c}_i} = K_{\text{chem}} + K_{\text{spatial}}$$

$$K_{\text{spatial}} = \int_0^{\infty} \frac{\partial r}{r} \int d\mathbf{x}\, k(r,\mathbf{x})$$

$$k(r,\mathbf{x}) = r^2 \sum_i \tilde{c}_i(r,\mathbf{x}) \left(\nabla \ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}} \right)^2 = r^2 \sum_i \frac{(\nabla \tilde{c}_i(r,\mathbf{x}))^2}{\tilde{c}_i(r,\mathbf{x})} = \left(-r\frac{\partial}{\partial r} + r^2 \nabla^2 \right) \sum_i \tilde{c}_i(r,\mathbf{x}) \ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}$$

$$\dot{c}_i(\mathbf{x},t) = \frac{d}{dt} c_i(\mathbf{x},t) = D_i \nabla^2 c_i(\mathbf{x},t) + F_i(\mathbf{c}(\mathbf{x},t)) + b_i(c_{i,\text{res}} - c_i(\mathbf{x},t))$$

$$\sigma(\mathbf{x},t) = \sum_i \left(D_i \frac{(\nabla c_i(\mathbf{x},t))^2}{c_i(\mathbf{x},t)} - F_i(\mathbf{c}(\mathbf{x},t)) \ln\frac{c_i(\mathbf{x},t)}{c_{i0}} \right)$$

$$j_r(r,\mathbf{x},t) = \sum_i \left(D_i \frac{(\nabla \tilde{c}_i(r,\mathbf{x},t))^2}{\tilde{c}_i(r,\mathbf{x},t)} - \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) \ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}} \right),\ \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) = \exp(\tfrac{r^2}{2}\nabla^2) F_i(\mathbf{c}(\mathbf{x},t))$$

$$\mathbf{j}(r,\mathbf{x},t) = -r^2 \nabla \sum_i \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) \ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}},\quad J(r,x,t) = -\sum_i b_i(\tilde{c}_i + c_{i,\text{res}})[r\nabla \ln \tilde{c}_i]^2$$

$$\dot{k}(r,\mathbf{x},t) = r\frac{\partial}{\partial r} j_r(r,\mathbf{x},t) - \nabla \cdot \mathbf{j}(r,\mathbf{x},t) + J(r,x,t)$$

Chaos and information

$$\int dx \mu(x) \varphi(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \varphi(f^k(x(0))),$$

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln|f'(x(k))| = \int dx \mu(x) \ln|f'(x)|$$

$$h(\mu,\mathbf{A}) = \lim_{n\to\infty} \frac{1}{n} H(B^{(n)}) = \lim_{n\to\infty} \left(H(B^{(n+1)}) - H(B^{(n)}) \right),\quad s_\mu = \lim_{diam(\mathbf{A})\to 0} h(\mu,\mathbf{A}),\quad s_\mu = \lambda$$