TIF 150: Information theory for complex systems

Time: March 12, 2010, 14.00 – 18.00.
Allowed material: Calculator (type approved according to Chalmers rules).
Teacher and examiner: Kristian Lindgren (772 3131)

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, and 42p for 5.
ECTS grades: 25p for E, 28p for D, 34p for C, 38p for B, 42p for A.
The results will be available on March 26.

1. Balance measurements.
Assume you have 6 balls of which 2 are slightly heavier than the others, with an additional weight of 1% and 2%, respectively. You have at your disposal three balance measurements to find out which are the two heavy ones. Find such a procedure by using an information-theoretic approach. Answer also the following questions:

(a) What is the initial uncertainty (entropy)? Compare this with how much information you could gain from three ideal measurements.
(b) Is there a procedure that always determines the two heavy ones and sorts out which is the heaviest among the two?

(9 p)

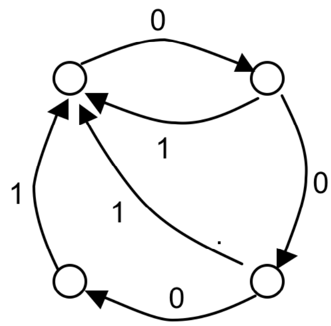
2. An equilibrium spin system. Consider a one-dimensional lattice system where each site can be occupied by a particle, either ⊗ or ⊕, or be empty –

...	⊗	⊗	–	–	⊕	⊕	⊕	⊗	⊕	–	⊗	⊗	–	...

There is a local contribution to the energy, being $-J < 0$ when two neighbouring particles are of the same type, and $+J$ when they are different. There is no interaction energy when the neighbouring site is empty. The average particle density is 1/3 for each particle type. If the average energy is u, what is the equilibrium distribution? Or, in other words, what are the probabilities over sequences of symbols that characterise the system? (You may give the answer as a function of temperature instead of energy.) What are the probabilities in the limit of zero temperature?

(9 p)

3. **CA information.**
Consider a one-dimensional cellular automaton given by elementary rule R124, i.e., neighbourhoods 111, 001 and 000 map to 0, but all others to 1. Let the initial state be characterized by the following finite state automaton (where it is assumed that if two arcs leave the same node they have equal probabilities)

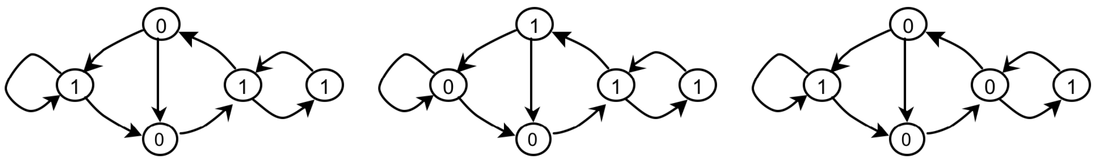


(a) What is the initial entropy (at $t = 0$)? Derive the finite state automaton that characterizes the CA state after one time step ($t = 1$). What is the entropy at this time?

(b) What is the entropy at $t = 2$?

(10 p)

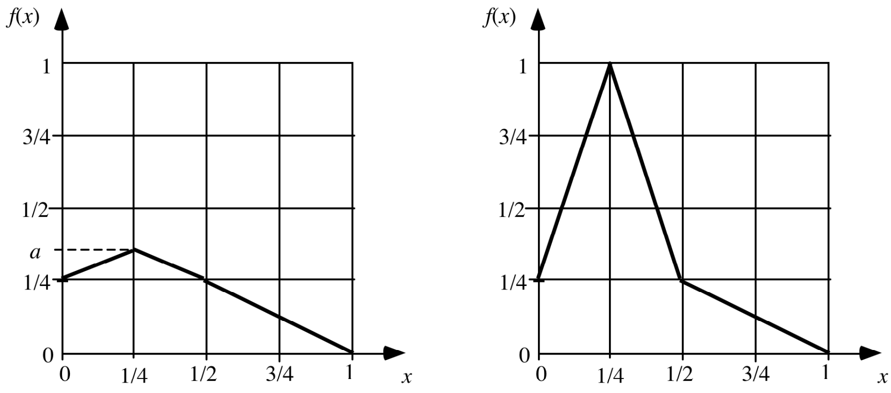
4. **Correlation complexity.** Below are three hidden Markov models, (a, b, and c). When two arcs leave a node it is assumed that they have the same probability. Determine the correlation complexity η for all three.



(a) (b) (c)

(10 p)

5. Chaos and information. Let a piecewise linear map $f(x)$ be defined by the figure below, where $1/4 < a < 1$, and where the mapping is determined by $f(0) = f(1/2) = 1/4$, $f(1/4) = a$, and $f(1) = 0$.



(i) (ii)

Consider the dynamical system

$$x_{t+1} = f(x_t) .$$

(i) Start with a close to 1/4 and investigate how the dynamics changes when a is increased. Determine whether there is a stable fixed point, stable periodic orbit, or chaos. What are the critical values for a (for which there is a change in dynamical characteristics)?

(ii) Suppose now that $a = 1$. Determine the invariant measure that characterizes the chaotic behaviour, and calculate the Lyapunov exponent λ. Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

(12 p)

Information theory for complex systems – useful equations

Basic quantities

$$I(p) = \log\frac{1}{p} \qquad S[P] = \sum_{i=1}^{n} p_i \log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right] = \sum_{i=1}^{n} p_i \log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k + 1$ constraints) using the Lagrangian L

$$L\left(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu\right) = S[P] + \sum_{k=1}^{r} \lambda_k \left(F_k - \sum_{i=1}^{n} p_i f_k(i)\right) + (\mu - 1)\left(1 - \sum_{i=1}^{n} p_i\right) ,$$

$$p_j = \exp\left(-\mu - \sum_{k=1}^{r} \lambda_k f_k(j)\right), \quad \mu(\lambda) = \ln\sum_{j=1}^{n} \exp(-\sum_{k} \lambda_k f_k(j)), \quad \frac{\partial\mu(\lambda)}{\partial\lambda_k} = -F_k$$

Symbol sequences

$$s = \lim_{n\to\infty} \sum_{x_1\ldots x_{n-1}} p(x_1\ldots x_{n-1}) \sum_{x_n} p(x_n \mid x_1\ldots x_{n-1}) \log\frac{1}{p(x_n \mid x_1\ldots x_{n-1})} =$$

$$= \lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \Delta S_\infty = \lim_{n\to\infty}\frac{1}{n} S_n$$

$$k_1 = \log\nu - S_1, \qquad k_n = -S_n + 2S_{n-1} - S_{n-2} = -\Delta S_n + \Delta S_{n-1} = -\Delta^2 S_n \quad (n = 2,3,\ldots)$$

$$S_{\text{tot}} = \log\nu = \sum_{m=1}^{\infty} k_m + s$$

$$\eta = \sum_{m=1}^{\infty} (m-1) k_m = \lim_{m\to\infty}\lim_{n\to\infty} \sum_{\sigma_m} p(\sigma_m) \sum_{\tau_n} p(\tau_n \mid \sigma_m) \log\frac{p(\tau_n \mid \sigma_m)}{p(\tau_n)} = \lim_{m\to\infty}(S_m - m\, s)$$

Geometric information theory

$$p(r;x) = \frac{1}{\sqrt{2\pi}\, r} \int_{-\infty}^{\infty} dw\, e^{-w^2/2r^2}\, p(x-w) \quad , \qquad \left(-r\frac{\partial}{\partial r} + r^2\frac{d^2}{dx^2}\right) p(r;x) = 0$$

$$p_{\text{Gaussian}}(r;x) = \frac{1}{\sqrt{2\pi}\sqrt{b^2 + r^2}} \exp\left(-\frac{x^2}{2(b^2 + r^2)}\right)$$

$$K[p_0;p] = \int dx\, p(x) \ln\frac{p(x)}{p_0(x)} = \int_0^{\infty} \frac{dr}{r} \int dx\, k(r,x), \qquad k(r,x) = r^2 p(r;x)\left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r) = D_{\text{E}} - r\frac{\partial}{\partial r} \int d\mathbf{x}\, p(r;\mathbf{x}) \ln\frac{1}{p(r;\mathbf{x})}$$

Chemical systems and information flow

$$E = k_{\text{B}} T_0 \frac{N}{V} K, \qquad K = \int_V d\mathbf{x}\, K[c_0;c(\mathbf{x})] = \int_V d\mathbf{x} \sum_{i=1}^{M} c_i(\mathbf{x}) \ln\frac{c_i(\mathbf{x})}{c_{i0}}, \quad \sum_{i=1}^{M} c_i(\mathbf{x},t) = 1$$

$$K = V\sum_{i=1}^{M}\bar{c}_i \ln\frac{\bar{c}_i}{c_{i0}} + \int_V d\mathbf{x}\sum_{i=1}^{M} c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{\bar{c}_i} = K_{\text{chem}} + K_{\text{spatial}}$$

$$K_{\text{spatial}} = \int_0^\infty \frac{\partial r}{r}\int d\mathbf{x}\, k(r,\mathbf{x})$$

$$k(r,\mathbf{x}) = r^2\sum_i \tilde{c}_i(r,\mathbf{x})\left(\nabla \ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}\right)^2 = r^2\sum_i \frac{(\nabla\tilde{c}_i(r,\mathbf{x}))^2}{\tilde{c}_i(r,\mathbf{x})} = \left(-r\frac{\partial}{\partial r} + r^2\nabla^2\right)\sum_i \tilde{c}_i(r,\mathbf{x})\ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}$$

$$\dot{c}_i(\mathbf{x},t) = \frac{d}{dt}c_i(\mathbf{x},t) = D_i\nabla^2 c_i(\mathbf{x},t) + F_i(\mathbf{c}(\mathbf{x},t)) + b_i(c_{i,\text{res}} - c_i(\mathbf{x},t))$$

$$\sigma(\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla c_i(\mathbf{x},t))^2}{c_i(\mathbf{x},t)} - F_i(\mathbf{c}(\mathbf{x},t))\ln\frac{c_i(\mathbf{x},t)}{c_{i0}}\right)$$

$$j_r(r,\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla\tilde{c}_i(r,\mathbf{x},t))^2}{\tilde{c}_i(r,\mathbf{x},t)} - \tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}}\right),\ \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) = \exp(\tfrac{r^2}{2}\nabla^2)F_i(\mathbf{c}(\mathbf{x},t))$$

$$\mathbf{j}(r,\mathbf{x},t) = -r^2\nabla\sum_i \tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}},\quad J(r,x,t) = -\sum_i b_i(\tilde{c}_i + c_{i,\text{res}})[r\nabla\ln\tilde{c}_i]^2$$

$$\dot{k}(r,\mathbf{x},t) = r\frac{\partial}{\partial r}j_r(r,\mathbf{x},t) - \nabla\cdot\mathbf{j}(r,\mathbf{x},t) + J(r,x,t)$$

<u>Chaos and information</u>

$$\int dx\mu(x)\varphi(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}\varphi(f^k(x(0))),$$

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}\ln|f'(x(k))| = \int dx\mu(x)\ln|f'(x)|$$

$$h(\mu,\mathbf{A}) = \lim_{n\to\infty}\frac{1}{n}H(B^{(n)}) = \lim_{n\to\infty}\left(H(B^{(n+1)}) - H(B^{(n)})\right),\quad s_\mu = \lim_{diam(\mathbf{A})\to 0} h(\mu,\mathbf{A}),\quad s_\mu = \lambda$$