TIF 150: Information theory for complex systems

Time: March 13, 2009, 14.00 – 18.00.
Allowed material: Calculator (type approved according to Chalmers rules).
Teacher: Johan Nyström (7723130)
Examiner: Kristian Lindgren

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, and 42p for 5.
ECTS grades: 25p for E, 28p for D, 34p for C, 38p for B, 42p for A.
The results will be available on March 27.

1. **Balance measurements.**
   Assume you have 5 balls of which 3 are of equal weight but different from the remaining 2 of which one is 1% heavier and one is 1% lighter. You have at your disposal three balance measurments to find out which are the normal, the light, and the heavy ones. Find such a procedure by using an information-theoretic approach. Answer the following questions.

   (a) What is the initial uncertainty (entropy)? Compare this with how much information you could gain from three ideal measurements.
   (b) What is the worst case outcome of your first measurement? Quantify this by calculating the uncertainty that remains.
   (c) For the complete procedure, identifying all balls in three measurements, it would be enough if you show that the worst case always works. (Worst case means that you get the outcome of a balance measurement where there are the most possibilities left, i.e., the remaing entropy is the highest.)

   (9 p)

2. **An equilibrium spin system.** Consider a one-dimensional infinite sequence of states, A, B, and C, where a microstate can be illustrated as

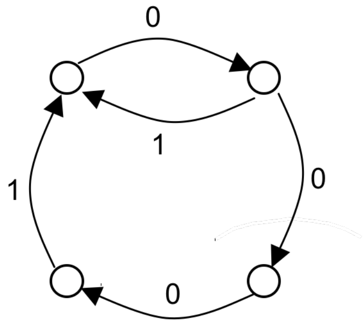
…	A	B	C	A	C	B	A	B	A	B	C	C	B	…

   There is a local contribution to the energy, being $-J$ when two neighbouring states are different, and 0 when they are the same. If the average energy is u, what is the equilibrium distribution? Or, in other words, what are the probabilities over sequences of symbols that characterise the system? (You may give the answer as a function of temperature instead of energy.) What are the probabilities in the limit of zero temperature?

   (9 p)

3. CA information.

Consider a one-dimensional cellular automaton given by elementary rule R252, i.e., neighbourhoods 000 and 001 map to 0, but all others to 1. Let the initial state be characterised by the following finite state automaton (where it is assumed that the arcs leaving the top right node have equal probabilities)



(a) What is the initial entropy (at $t = 0$)? Derive the finite state automaton that characterizes the CA state after one time step ($t = 1$). What is the entropy at this time?

(b) What is the entropy at $t = 2$ and $t = 3$?

(10 p)

4. Change of chemical information in an open system.

Show that, for an open chemical system (in contact with a reservoir), the change of chemical information is determined by the information flow towards smaller length scales j_r at the worst level of resolution ($r \to \infty$), and the *inflow of chemical information* j_{chem}, where we define

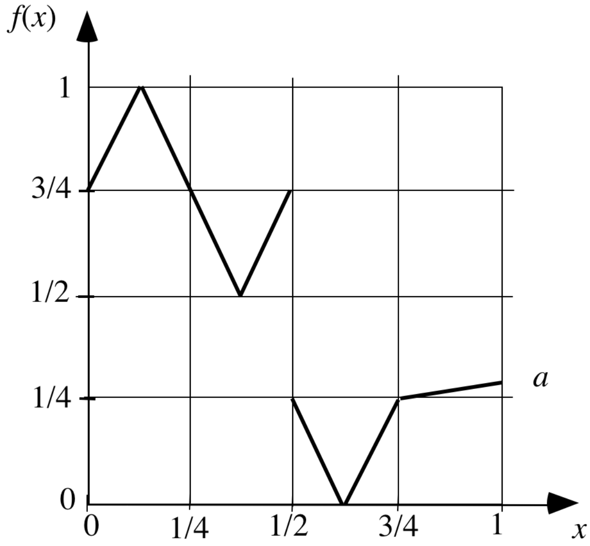
$$j_{\text{chem}}(t) = \sum_i b_i (c_{i,\text{ res}} - \bar{c}_i(t)) \ln \frac{\bar{c}_i(t)}{c_{i0}}$$

where $c_{i,res}$ is the concentration in the reservoir.

Assume, as we have done in the course, that we have reaction-diffusion dynamics (see equations sheet) with periodic boundary conditions at x=0 and x=L, and that concentrations are normalised at every position, i.e., $\Sigma_i \, c_i(x, t) = 1$.

(10 p)

5. Chaos and information. Let a piecewise linear mapping $f(x)$ be defined by the figure below, where $1/4 < a < 1$, and where the mapping is determined by $f(0) = f(1/4) = f(1/2) = 3/4$, $f(1/8) = 1$, $f(3/8) = 1/2$, $f(1/2^+) = f(3/4) = 1/4$, $f(5/8) = 0$, and $f(1) = a$.



Consider the dynamical system

$$x_{t+1} = f(x_t) \ .$$

Starting with a close to 1/4 and increasing that value, at what value of a does the system become chaotic? What is the dynamic behaviour of the system for a lower than this value?

Suppose now that $a = 3/4$. Determine the invariant measure that characterizes the chaotic behaviour, and calculate the Lyapunov exponent λ. Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

If you know that the system is in the region $x < 1/4$ at time t, how much information do you get if you observe the system in the same region again at time $t+4$?

(12 p)

Information theory for complex systems – useful equations

Basic quantities

$$I(p) = \log\frac{1}{p} \qquad S[P] = \sum_{i=1}^{n} p_i \log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right] = \sum_{i=1}^{n} p_i \log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k + 1$ constraints) using the Lagrangian L

$$L\left(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu\right) = S[P] + \sum_{k=1}^{r} \lambda_k \left(F_k - \sum_{i=1}^{n} p_i f_k(i)\right) + \left(\mu - 1\right)\left(1 - \sum_{i=1}^{n} p_i\right),$$

$$p_j = \exp\left(-\mu - \sum_{k=1}^{r} \lambda_k f_k(j)\right), \quad \mu(\lambda) = \ln\sum_{j=1}^{n}\exp(-\sum_{k}\lambda_k f_k(j)), \quad \frac{\partial\mu(\lambda)}{\partial\lambda_k} = -F_k$$

Symbol sequences

$$s = \lim_{n\to\infty} \sum_{x_1\ldots x_{n-1}} p(x_1\ldots x_{n-1}) \sum_{x_n} p(x_n \mid x_1\ldots x_{n-1}) \log\frac{1}{p(x_n \mid x_1\ldots x_{n-1})} =$$

$$= \lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \Delta S_\infty = \lim_{n\to\infty}\frac{1}{n} S_n$$

$$k_1 = \log\nu - S_1, \qquad k_n = -S_n + 2S_{n-1} - S_{n-2} = -\Delta S_n + \Delta S_{n-1} = -\Delta^2 S_n \quad (n = 2,3,\ldots)$$

$$S_{\text{tot}} = \log\nu = \sum_{m=1}^{\infty} k_m + s$$

$$\eta = \sum_{m=1}^{\infty}(m-1)\,k_m = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{\sigma_m} p(\sigma_m)\sum_{\tau_n} p(\tau_n \mid \sigma_m)\log\frac{p(\tau_n \mid \sigma_m)}{p(\tau_n)} = \lim_{m\to\infty}(S_m - m\,s)$$

Geometric information theory

$$p(r;x) = \frac{1}{\sqrt{2\pi}\,r}\int_{-\infty}^{\infty} dw\, e^{-w^2/2r^2}\, p(x-w) \quad , \qquad \left(-r\frac{\partial}{\partial r} + r^2\frac{d^2}{dx^2}\right)p(r;x) = 0$$

$$p_{\text{Gaussian}}(r;x) = \frac{1}{\sqrt{2\pi}\sqrt{b^2 + r^2}}\exp\left(-\frac{x^2}{2(b^2 + r^2)}\right)$$

$$K[p_0;p] = \int dx\, p(x)\ln\frac{p(x)}{p_0(x)} = \int_0^{\infty}\frac{dr}{r}\int dx\, k(r,x), \qquad k(r,x) = r^2 p(r;x)\left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r) = D_{\text{E}} - r\frac{\partial}{\partial r}\int d\mathbf{x}\, p(r;\mathbf{x})\ln\frac{1}{p(r;\mathbf{x})}$$

Chemical systems and information flow

$$E = k_{\text{B}}T_0\frac{N}{V}K, \qquad K = \int_V d\mathbf{x}\, K[c_0;c(\mathbf{x})] = \int_V d\mathbf{x}\sum_{i=1}^{M} c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{c_{i0}}, \quad \sum_{i=1}^{M} c_i(\mathbf{x},t) = 1$$

$$K = V\sum_{i=1}^{M}\bar{c}_i \ln\frac{\bar{c}_i}{c_{i0}} + \int_V d\mathbf{x}\sum_{i=1}^{M} c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{\bar{c}_i} = K_{\text{chem}} + K_{\text{spatial}}$$

$$K_{\text{spatial}} = \int_0^\infty \frac{\partial r}{r}\int d\mathbf{x}\, k(r,\mathbf{x})$$

$$k(r,\mathbf{x}) = r^2\sum_i \tilde{c}_i(r,\mathbf{x})\left(\nabla \ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}\right)^2 = r^2\sum_i \frac{(\nabla\tilde{c}_i(r,\mathbf{x}))^2}{\tilde{c}_i(r,\mathbf{x})} = \left(-r\frac{\partial}{\partial r} + r^2\nabla^2\right)\sum_i \tilde{c}_i(r,\mathbf{x})\ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}$$

$$\dot{c}_i(\mathbf{x},t) = \frac{d}{dt}c_i(\mathbf{x},t) = D_i\nabla^2 c_i(\mathbf{x},t) + F_i(\mathbf{c}(\mathbf{x},t)) + b_i(c_{i,\,\text{res}} - c_i(\mathbf{x},t))$$

$$\sigma(\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla c_i(\mathbf{x},t))^2}{c_i(\mathbf{x},t)} - F_i(\mathbf{c}(\mathbf{x},t))\ln\frac{c_i(\mathbf{x},t)}{c_{i0}}\right)$$

$$j_r(r,\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla\tilde{c}_i(r,\mathbf{x},t))^2}{\tilde{c}_i(r,\mathbf{x},t)} - \tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}}\right),\ \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) = \exp(\tfrac{r^2}{2}\nabla^2)F_i(\mathbf{c}(\mathbf{x},t))$$

$$\mathbf{j}(r,\mathbf{x},t) = -r^2\nabla\sum_i \tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}},\quad J(r,x,t) = -\sum_i b_i(\tilde{c}_i + c_{i,\,\text{res}})[r\nabla\ln\tilde{c}_i]^2$$

$$\dot{k}(r,\mathbf{x},t) = r\frac{\partial}{\partial r}j_r(r,\mathbf{x},t) - \nabla\cdot\mathbf{j}(r,\mathbf{x},t) + J(r,x,t)$$

<u>Chaos and information</u>

$$\int dx\mu(x)\varphi(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}\varphi(f^k(x(0))),$$

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}\ln|f'(x(k))| = \int dx\mu(x)\ln|f'(x)|$$

$$h(\mu,\mathbf{A}) = \lim_{n\to\infty}\frac{1}{n}H(B^{(n)}) = \lim_{n\to\infty}\left(H(B^{(n+1)}) - H(B^{(n)})\right),\quad s_\mu = \lim_{diam(\mathbf{A})\to 0}h(\mu,\mathbf{A}),\quad s_\mu = \lambda$$

<u>Algorithmic information</u>

$$H_U(\alpha_m) = \min_{U(P,X)=\alpha_m} l(P) + l(X)$$

$$L_U(\omega_d) = \sum_{\gamma\in\omega_d}\min\left(H_U(\gamma), l(\gamma)\right),\quad L_U^{(d)}(\alpha_m) = \min_{\omega_d}\left(L_U(\omega_d)\right)$$

$$C_U^{(d)}(\alpha_m) = L_U^{(d-1)}(\alpha_m) - L_U^{(d)}(\alpha_m)$$

$$m = \sum_{d=2}^{m} C_U^{(d)}(\alpha_m) + L_U^{(m)}(\alpha_m)$$