Information theory for complex systems – FFR050

Time: March 16, 2005, 14.00 – 18.00.
Allowed material: anything except other person.
Examiner/teacher: Kristian Lindgren, tel. 772 3131.

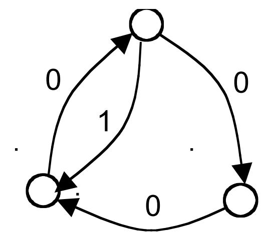
The results are planned to be posted outside my office on Wednesday 13.00 (March 23), where we may also discuss the solutions. (You may also ask about your result by email.)

1. **Mathematical requirements for entropy.** The quantity S[P] that we have used for entropy of a distribution P = $\{p_1, p_2, \ldots, p_n\}$ over microstates $(1, 2, \ldots, n)$ is the only quantity fulfilling the following four conditions: (i) S is symmetric with respect to the probabilities, (ii) S is a continuous function of the probabilities, (iii) The information obtained when one gets to know the outcome of two equally probable events is 1 bit, and finally (iv):
The expected gain of information is the same for (I) an immediate observation of the microstate as for (II) a two step observation in which one distinguishes between, say, state 1 and state 2 only if a first observation rejects the other states.

   Express the last condition (iv) in mathematical terms, i.e., express the entropy S as a sum of two entropies from two measurements as described in (II). Show that this expression holds for the Shannon definition of entropy: $S = \Sigma\, p_k \log(1/ p_k)$.

   (8 p)

2. **CA entropy.**
Consider a one-dimensional cellular automaton given by elementary rule 71 (where configurations 110, 010, 001, and 000 result in a 1 and the rest give 0). Let the initial state be characterised by the following finite state automaton



   where the probabilities for choosing an arc is always the same (1/2) if there is a choice. What is the initial entropy (t = 0), and what is the entropy at t = 1 and t = 2?

   (12 p)

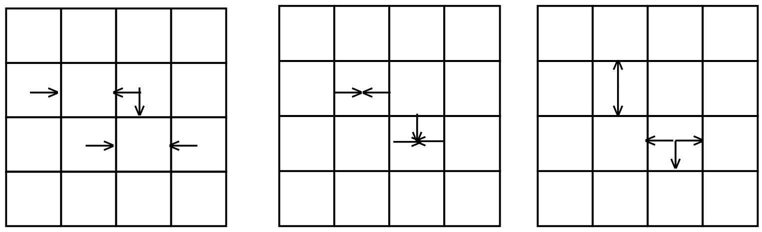
3. Happy agents. Consider an infinite one-dimensional lattice system (of cells) in which each cell may be inhabited by a red (R) individual, a blue (B) individual, a pair (RB of agents with different colours), or the cell may be empty. Assume equal densities of individuals R and B of 1/4 each. Each individual has a happiness level being the sum of the happiness from the relation with the closest neighbours. If there is a **pair in a cell** the happiness of that cell is $4H$ (with H being a positive "happiness" constant), and then there is no contribution from the neighbouring cells. If there is a **single individual in a cell**, the happiness of that individual get a contribution of $H/2$ from each single living neighbour of opposite colour (to the left and to the right, but no contribution from neighbour pairs). Empty cells do not contribute to happiness.

If you know that the average happiness is w, how would you guess that the system looks like in equilibrium, using information-theoretic arguments. <u>You may answer in terms of a set of equations that you need *not* solve.</u> Discuss what happends if the "temperature" is low (limit of zero temperature), with the interpretation that a low temperature corresponds to a high "happiness". What is the entropy in this limit?

(If you prefer, all this can be thought of as molecules A and B that may aggregate to a larger molecules AB, with the interpretation of H as a negative interaction energy constant.)

(9p)

4. Lattice gas entropy. Consider an infinite 2-dimensional lattice gas constructed in the following way. The space is a square lattice and in each cell up to four particles may be present (one in each direction). The system evolves in discrete time, and in each time step there is movement and collision. Particles move from one cell to the next according to the direction of the particle. A collision occurs if and only if exactly two particles enter a cell with opposite directions, and then the direction of these particles are shifted so that they leave perpendicular to their initial directions. The two processes in a single time step is illustrated in the following figure:



Suppose we have a system where we initially (at $t = 0$) have equal densities of the four particle directions ($\rho/4$ each with ρ being the overall particle density), but where particles initially are present only in cells where *all* particle directions are present. Assume that these cells, each containing four particles, are randomly distributed over the whole lattice.

Consider the entropy s of the spatial configuration of particles, based on the 2-dimensional block entropy

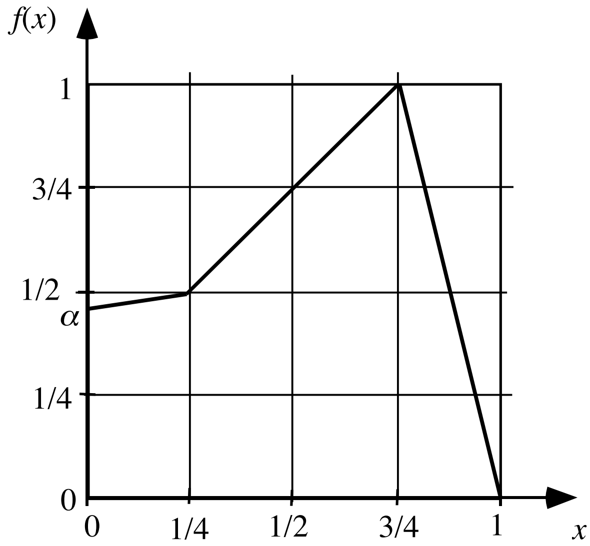
$$s = \lim_{m \to \infty} \frac{1}{m^2} S_{m \times m}$$

What is the entropy s at $t = 0$? How does the entropy change in the time evolution?

If one would estimate the entropy using a finite block size m after very long time T, with $T >> m$, what result should one expect? What is the explanation?

(9 p)

5. Chaos and information. Let a mapping $f(x)$ be defined by the figure below (so that $f(0) = \alpha, f(1/4) = 1/2, f(3/4) = 1, f(1) = 0$, with $0 \leq \alpha \leq 1/2$.



Consider the dynamical system

$$x_{t+1} = f(x_t) \ .$$

a) At which value of $\alpha < 1/2$ does the system become chaotic when decreasing from 1/2?

b) What is the behaviour for α close to 1/2 (above the critical value derived above). Describe qualitatively only.

Assume from now on that $\alpha = 0$.

c) Find the invariant measure μ that characterises the chaotic behaviour, and determine the corresponding Lyapunov exponent λ by using that measure. Find a partition that has a symbolic dynamics with a measure entropy s_μ that equals the Lyapunov exponent (as one should expect).

c) Suppose that we at a certain time t observe the system in the region given by $x > 3/4$. If we find the system in this region again three time steps later (at $t + 3$), how much information do we gain by this observation?

(12 p)

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(1) $S[P] = \sum_i p_i \log \frac{1}{p_i} = S[p_1+p_2, p_3, \ldots, p_n] + (p_1+p_2) \cdot S\left[\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right]$

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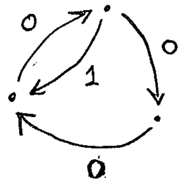
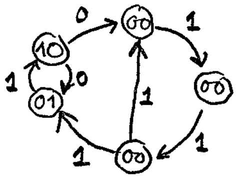
(2)



stationary distribution $\{\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\}$

$\rightarrow\ s = \frac{2}{5} \ln 2$

equiv. autom. with $t=0$ in nodes and $t=1$ on arcs



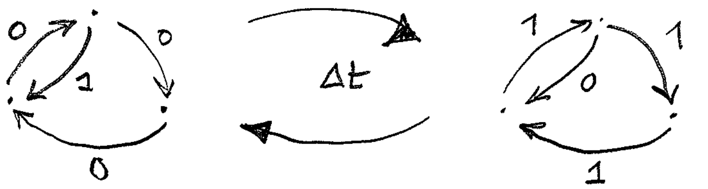
The rule from $t=0$ to $t=1$ is left-side almost reversible since pairs of 1's are not present at $t=0$.

From $t=1$ to $t=2$, since pairs of 0's are not present at $t=1$, the rule is right-side almost reversible.

Implies

$s(t=0) = s(t=1) = s(t=2) = \frac{2}{5} \ln 2$

In detail you may find that:



at $t=2n$ at $t=2n+1$

(Other ways to solve also possible, e.g. coding of $10 \rightarrow A$, $000 \rightarrow B$, etc.)

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(3) Nearest neighbour interaction only $\longrightarrow s = S_2 - S_1 \rightarrow$ pairs of cells necessary in description

Cell states: [R/B], [R], [B], [–]

Pair config	happiness between cells	multipl.	prob.
[R/B][R/B]	–	1	p_0
[R/B][?] ← R, B, –	–	2 × 3	p_1
[R][B]	$2 \times \frac{H}{2} = H$	2	p_2
[B][B] } [R][R]	–	2	p_3
[B/R][–]	–	2 × 2	p_4
[–][–]	–	1	p_5

$P([R/B]) = p_0 + 3p_1$

$P([B]) = P([R]) = p_1 + p_2 + p_3 + p_4$

$P([–]) = p_1 + 2p_4 + p_5$

(i) $\rho_R = \frac{1}{4} \rightarrow \frac{1}{4} = (p_0 + 3p_1) + (p_1 + p_2 + p_3 + p_4)$

(ii) $1 = p_0 + 6p_1 + 2p_2 + 2p_3 + 4p_4 + p_5$

(iii) $W = P([R/B]) \cdot 4H + 2P([R][B]) \cdot \frac{H}{2} \cdot 2 =$

$= (p_0 + 3p_1) 4H + 2p_2 \cdot H$

continued →

③ (contin.) Problem formulation to find equilibr. state :

max s under constraints (i), (ii), and (iii)

with $s = S_2 - S_1$ and

$$S_1 = S\left[\, p\left(\begin{smallmatrix}R\\B\end{smallmatrix}\right),\ p(R),\ p(B),\ p(-)\right]$$

$$S_2 = \sum_{k=1}^{5} m_k\, p_k \ln \frac{1}{p_k}$$, with p_k and multiplicity m_k from table of config.

(max-problem can be solved by Lagrangian

$$L = s - \beta\left(w - (p_0+3p_1)4H - 2p_2 \cdot H\right) + \lambda\left(\tfrac{1}{4} - (p_0+3p_1) - (p_1+p_2+p_3+p_4)\right)$$
$$+ \mu\left(1 - \sum_k m_k p_k\right)$$

with β, λ, and μ as Lagrangian variables.)

At maximum happiness all individuals form pairs $\boxed{\begin{smallmatrix}R\\B\end{smallmatrix}}$
there should be no correlation between cell at max-s.

Then $p\left(\boxed{\begin{smallmatrix}R\\B\end{smallmatrix}}\right) = \frac{1}{4}$ $\quad p(\boxed{-}) = \frac{3}{4}$ and $s = S_1 = \frac{1}{4}\ln 4 + \frac{3}{4}\ln\frac{4}{3}$
$= \ln 4 - \frac{3}{4}\ln 3$

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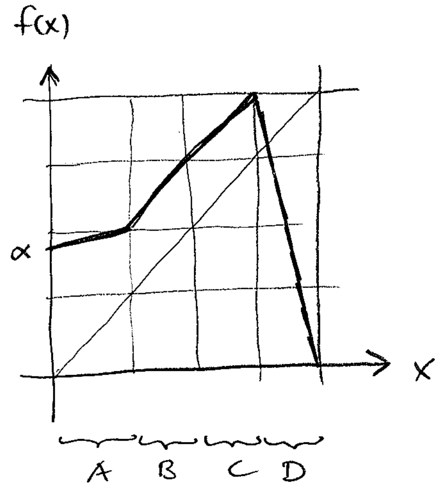
④ reversible CA, s is conserved!

with initial state given by $p\left(\boxed{\leftrightarrow\updownarrow}\right) = \frac{\rho}{4}$ and $p(\square) = 1 - \frac{\rho}{4}$
(and no correlations between cells)
we have $s = S_1 = S\left[\frac{\rho}{4},\ 1 - \frac{\rho}{4}\right]$

If we measure, at time t, the entropy based on statistics on blocks of size $m \times m$ (with $t \gg m$), we do not capture the correlations that contain the information of the initial state, and the estimated entropy will be larger. Typically, we should expect that any configuration having k particles within a cell, occur with probability $P_k = \left(\frac{\rho}{4}\right)^k \left(1 - \frac{\rho}{4}\right)^{4-k}$, and that the estimated entropy will be $\tilde{s} = \sum_k \binom{4}{k} P_k \ln \frac{1}{P_k}$.

(5)



if $\frac{1}{2}-\frac{1}{16} \le \alpha \le \frac{1}{2}$: A → B → C → [D] (→ A); [D] only maps to the right-hand part of D, when $\frac{1}{2}-\frac{1}{16} \le \alpha \le \frac{1}{2}$

then $\mu(A)=\mu(B)=\mu(C)=\mu(D)=\frac{1}{4}$ and $\lambda = \frac{1}{4}\ln|f'(A)| + 0 + 0 + \frac{1}{4}\ln 4$

with α in this interval $|f'(A)| \le \frac{1}{4} \rightarrow \lambda \le 0$

b/ → stable periodic orbit for $\alpha > \frac{1}{2}-\frac{1}{16}$

When α goes below (slightly) $\frac{1}{2}-\frac{1}{16}$ the transitions looks like :

this additional link decreases the weight of A ensuring that $\lambda > 0$ when $\alpha < \frac{1}{2}-\frac{1}{16}$

a/ → chaos

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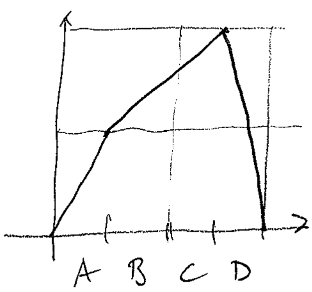
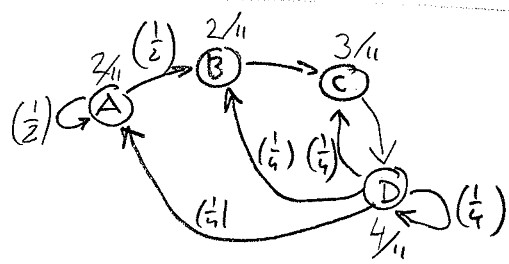
c/



$|f'(A)| = 2$
$|f'(B \cup C)| = 1$
$|f'(D)| = 4$



stationary/invariant μ : $\mu(A)=\mu(B)=\frac{2}{11}$, $\mu(C)=\frac{3}{11}$, $\mu(D)=\frac{4}{11}$

$\rightarrow \lambda = \sum_x \mu(x)\ln|f'(x)| = \frac{2}{11}\ln 2 + \frac{4}{11}\ln 4 = \frac{10}{11}\ln 2$

$S_\mu = \mu(A)\, S[\frac{1}{2},\frac{1}{2}] + \mu(D)\, S[\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}] = \frac{10}{11}\ln 2$

d/ $P(D \text{ at } t+3 \mid D \text{ at } t) = P(DDDD)+P(DDCD)+P(DCDD)+P(DBCD) = \frac{1}{64}+\frac{1}{16}+\frac{1}{16}+\frac{1}{4} = \frac{25}{64}$

$\rightarrow I(P) = \ln\frac{1}{P} = \ln\frac{64}{25}$