Information theory for complex systems

Time: March 11, 2003, FL73-74, 13.00-18.30.
Allowed material: anything except other person.
Examiner/teacher: Kristian Lindgren (phone 031-772 3131)

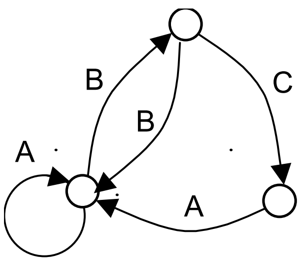
1. **Balance information.** You have a special balance device in the form of a triangle, so that you simultaneously can measure three sets of objects (and determine their weight order). Suppose that you have a number of N balls, so that all except one are of equal weights and the deviating one is known to be heavier. (a) Use an information-theoretic argument to determine the maximum number N for which you can always be sure to find the heavy ball in two measurements. (b) Now, suppose you do not know whether the deviating ball is heavier or lighter. Assuming 15 balls, what is the entropy? How many measurements are needed in this case? What is the maximum expected information gain one could get from a measurement in this case? Find a scheme that works!

(8 p)

2. **ABC-CA.** Assume that a three-state cellular automaton, with states A, B, and C, develops according to a rule that depends on the cell and its right neighbour, so that

t	AA	AB	AC	BA	BB	BC	CA	CB	CC
$t+1$	A	B	B	B	C	A	B	B	B

Let the initial state be characterised by the following finite state automaton



where the probabilities for choosing an arc is always the same (1/2) if there is a choice. What is the initial entropy ($t = 0$), and what is the entropy at $t = 1$ and $t = 2$?

(12 p)

3. **Familiar neighbourhood.** Consider an infinite one-dimensional lattice system (of cells) in which each cell is inhabited by either a red (R) or a blue (B) individual. These individuals are satisfied as soon as at least one of their neighbours is of their own kind, but unsatisfied otherwise. Assume that there is an equal number of As and Bs, and suppose that an unsatisfied individual contributes with a quantity J to the global irritation level that is known to be u (in average per cell). Derive the equations that can be used to determine the probabilistic description of this system. (You do not have to solve them.) Instead of using the the irritation level u, one can use the corresponding Lagrangian parameter β that can be viewed as an "inverse irritation temperature" as a different characterisation of level of unsatisfaction. What happens when this temperature approaches zero ($\beta \rightarrow \infty$)? Derive the probabilities in this particular case.

(10 p)

4. **Lattice gas.** A "reactive lattice gas" is a two-dimensional cellular automaton that consists of a lattice of square cells. Each cell may contain up to four different particles each of them having unit speed with a certain direction. Particles may only take one of the four lattice directions pointing to a neighbouring cell, and there may only be one particle in each direction. Particles may be of three molecular types: A and B are reacting and C is inert (does not react with the others). Particles keep their directions unless they collide and/or react.

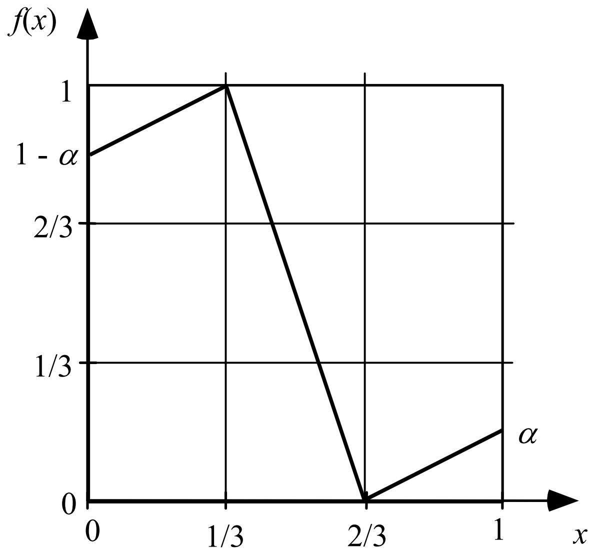
   In each time step of the lattice gas the following happens. (i) Particles move to adjacent cells according to the direction they are pointing to. If several particles enter the same cell, there may be a "collision" or a "reaction": (ii) A collision only occurs if exactly two particles of the same molecular type enter a cell and if they meet "head on". Such a "head on" collision results in the particles leaving the cell (in the next time step), both of them turned 90° and leaving in opposite directions. (iii) A reaction occurs if at least one molecule of each reacting type (A and B) enter a cell, and then B is transformed to C ($A+B \rightarrow A+C$), and all continue in their directions, respectively. In all other cases, when one or several particles enter a cell, they just pass through each other continuing in the same direction as before.

   Discuss qualitatively (and briefly) what is happening with the entropy in this system, in particular what role is played by the three different steps (i)-(iii) for the (possible) change in entropy (randomness).

   If the system is made "open" by allowing for an inflow of B molecules and an outflow of C molecules, the entropy may be affected. These flows are realised as random introduction and removal of B and C molecules, respectively. Discuss qualitatively (and briefly) whether this modification leads to an increase or a decrease in entropy.

(7 p)

5. Chaos and information. Let a mapping $f(x)$ be defined by the figure below, where $f(1/3) = 1, f(2/3) = 0, f(0) = 1-\alpha$, and $f(1) = \alpha$.



Consider the dynamical system

$$x_{t+1} = f(x_t) .$$

a) What is the behaviour for $\alpha = 1/6$? At which value of α does the system become chaotic?

Assume from now on that $\alpha = 2/3$.

b) Find the invariant measure μ that characterises the chaotic behaviour, and determine the corresponding Lyapunov exponent λ by using that measure. Find a partition that has a symbolic dynamics with a measure entropy s_μ that equals the Lyapunov exponent (as one should expect).

c) Suppose that we at a certain time t observe the system in the region given by $x < 1/3$. If we find the system in this region again two time steps later (at $t + 2$), how much information do we gain by this observation?

(13 p)

Answers or hints to problems of exam from March 2003.

1. Balance information.
   a) $N = 16$
   b) $S = \ln 30$
      2 measurements
      $I_{max} = \ln 7$
      Weigh 4+4+4 and 4 on the side.

2. Entropy is conserved in first time step; can be seen from observing the rule table, as crossing out neighbourhoods that do not occur leads to an almost reversible rule. But this does not work for the 2nd step since new neighbourhood combinations appear that makes such a simple approach impossible. Here it is not necessary to calculate s at t=2 (it was more tricky than I thought), but you should derive the automaton and argue that entropy decreases between t=1 and t=2.

3. Blocks of three cells need to be considered. Only blocks BRB and RBR contribute with J, and because of symmetry they are of equal probability, p_0. Symmetry also implies that blocks BBB and RRR can be assigned probability p_1, and blocks BBR, RRB, BRR, RBB probability p_2. Normalisation implies $1 = 2p_0 + 2p_1 + 4p_2$. To calculate s, we need S_3-S_2. Now for S_2, $P(BB)=P(RR) = p_1 + p_2$, and $P(BR)=P(RB)= p_0 + p_2$. Maximization of s is given by maximazing $L = S_3-S_2 + \beta(u - 2 J p_0) + \mu(1 - (2p_0 + 2p_1 + 4p_2))$. For the case of zero temperature (or $\beta \to \infty$), p0 = 0, and the entropy can be written
$s(p) = -(p \ln p + (1-p) \ln (1-p)) / (2(1+p))$, where $p = P(R|BB)$.
Max leads to $p^2 - 3 p + 1 = 0$, which determines the probabilities.

5. Chaos and information.
   a) Periodic, stable: $x_1 = 1/9$ and $x_2 = 8/9$.
   b) $A=\{x: 0 \leq x < 1/3 \}$, $B=\{x: 1/3 \leq x < 2/3 \}$, $C=\{x: 2/3 \leq x \leq 1 \}$,
      $\mu(A) = \mu(C) = 2/7$, $\mu(B) = 3/7$.
      $\lambda = 4/7 \ln 2 + 3/7 \ln 3$.
   c) $I(A_{t+2} | A_t) = - \ln P(A_{t+2} | A_t) = - \ln(1/4 + 1/6)$.