Reglerteknik ESS017
August 2015

This exam contains 11 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You are only allowed to use the formula sheet for this exam, and a calculator.

You are required to show your work on each problem on this exam.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive less credit.
- **Mysterious or unsupported answers will not receive full credit**, but an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- None of the proposed questions require extremely long computations. If you get caught in endless algebra, you have probably missed the simple way of doing it.
- The passing grade will a priori be given at 25 points. This limit may be lowered depending on the outcome of the exam.

Problem	Points	Score
1	9	
2	6	
3	10	
4	9	
5	11	
Total:	45	

Best of luck to all !!

1. Consider an electrical motor attached to a mass, described by the Ordinary Differential Equation (ODE)

$$J\dot{\omega} = -\xi\omega + K_{\mathrm{m}}u \tag{1}$$

where ω is the motor rotation speed in rad/s, $J = 1$ is the overall system inertia, $\xi = 0.1$ the friction, and $K_{\mathrm{m}} = 10$ the motor gain.

(a) (2 points) A P-controller is applied to the system in order to perform speed control. What controller gain should we use in order to get a phase margin of 100° ?

(b) (1 point) What if a phase margin of 80° is demanded ? Explain.

(c) (1 point) We now turn to performing position control. Write the transfer function describing the dynamics of the motor position in rad.

(d) (5 points) Design a controller performing position control for the motor. Select the adequate controller structure provided that no permanent disturbance is expected in the system. We would like to have a cross-over frequency of w rad/s and maximise the phase margin. We want the controller $F(s)$ to have a magnitude of no more than 20 db, i.e. $\|F(j\omega)\|_\infty \leq 20$ db.

Solution: The transfer function associated to (1) reads as:

$$G(s) = \frac{K_{\mathrm{m}}}{Js+\xi} = \frac{10}{s+0.1}$$

(a) A P-controller $F(s) = K_{\mathrm{P}}$ cannot modify the phase of the open-loop transfer function $F(s)G(s)$, which reads as:

$$\measuredangle F(j\omega)G(j\omega) = \measuredangle G(j\omega) = -\mathrm{a}\tan\left(\frac{\omega}{0.1}\right) = -\mathrm{a}\tan\left(10\omega\right)$$

hence in order to achieve a phase margin of 100°, the cross-over frequency ω_{c} must be selected such that $\measuredangle F(j\omega_{\mathrm{c}})G(j\omega_{\mathrm{c}}) = -180 + 100 = -80°$, hence:

$$\mathrm{a}\tan\left(10\omega_{\mathrm{c}}\right) = 80°, \qquad \text{i.e.} \qquad \omega_{\mathrm{c}} = \frac{1}{10}\tan\left(80°\right) = 0.57 \text{ rad/s}$$

Moreover, at ω_{c}, the open-loop transfer function magnitude must be - by definition - unitary, i.e.

$$|F(j\omega_{\mathrm{c}})G(j\omega_{\mathrm{c}})| = K_{\mathrm{P}}\frac{10}{(\omega_{\mathrm{c}}^2 + 0.1^2)^{\frac{1}{2}}} = 1$$

hence

$$K_{\mathrm{P}} = \frac{\left(\omega_{\mathrm{c}}^2 + 0.1^2\right)^{\frac{1}{2}}}{10} = 0.058$$

(b) Because $G(s)$ is an un-delayed, first order system, its phase never drops below 90°. Using a P-controller, the phase margin is always of at least 90°, such that the requirement of having a phase margin of 80° does not impose any limitation on the gain K_{P} of the P-controller. In theory, an arbitrarily large closed-loop bandwidth can be achieved.

(c) If working in terms of position, the output of the system is simply an integration of the motor speed, i.e.

$$\theta(t) = \theta(0) + \int_0^t \omega(\tau)d\tau \tag{2}$$

The transfer function describing the motor position then reads as:

$$G_{\text{P}}(s) = \frac{1}{s}G(s) = \frac{10}{s^2 + 0.1s} \equiv \frac{10}{s\,(s + 0.1)} \tag{3}$$

(d) We are dealing with a second-order system. In order to get good closed-loop performance, a PD controller ought to be used. An integrator is not mandatory since no permanent disturbance is expected. Since a limitation on the controller maximum gain is imposed, a "lead-lag" structure is required, e.g. in the form (this form varies in the literature):

$$F(s) = K_{\text{P}}\frac{\tau s + 1}{\beta\tau s + 1}$$

The gain limitation then requires that:

$$\|F(j\omega)\|_{\infty} = \frac{K_{\text{P}}}{\beta} \leq 20 \text{ db}$$

In order to maximise the phase margin, we need to minimize β and place the maximum phase of the PD-controller at $\omega_{\text{c}} = 2$ rad/s, i.e. we need to have

$$\omega_{\text{c}} = \frac{1}{\sqrt{\beta}\tau} = 2, \qquad \frac{K_{\text{P}}}{\beta} = 20 \text{ db} \equiv 10 \tag{4}$$

Finally, the magnitude of the open-loop transfer function at the cross-over frequency ω_{c} must be - by definition - unitary, i.e.

$$|F(j\omega_{\text{c}})G(j\omega_{\text{c}})| = 1 \tag{5}$$

such that

$$|F(j\omega_{\text{c}})| = \frac{K_{\text{P}}}{\sqrt{\beta}} = |G(j\omega_{\text{c}})|^{-1} = \left(\frac{10}{\omega_{\text{c}}\,|\omega_{\text{c}}j + 0.1|}\right)^{-1} = \frac{\omega_{\text{c}}\sqrt{\omega_{\text{c}}^2 + 10^{-2}}}{10} = 0.4 \tag{6}$$

We then have the three conditions:

$$K_{\text{P}} = 10\beta, \qquad K_{\text{P}} = 0.4\sqrt{\beta}, \qquad \tau = \frac{1}{\omega_{\text{c}}\sqrt{\beta}} \tag{7}$$

which take the solution:

$$\beta = 1.6 \cdot 10^{-3}, \qquad K_{\text{P}} = 1.6 \cdot 10^{-2}, \qquad \tau = 12.5 \tag{8}$$

2. Consider the state-space representation

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{9a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \tag{9b}$$

(a) (3 points) Is the system observable ? Is it controllable ?

(b) (3 points) Provide an explanation (beyond the mathematic conditions we use to verify them) of the actual meaning of these terms. Be as specific and clear as possible in your explanation. Why are these notions important ?

Solution:

(a) The controllability matrix for (9) reads as:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

One can visually assess that this matrix is full rank, or alternatively compute its determinant $\det(\mathcal{C}) = 1$. The system is therefore controllable. Similarly, the observability matrix of (9) reads as:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, by simple inspection one can assess that $\mathcal{O}$ is rank-deficient, or alternatively compute $\det(\mathcal{O}) = 0$. The system is therefore not observable.

(b) A system is controllable if for any initial condition, all states can be driven to the origin in finite time by an adequate choice of control input $u(t)$. A lack of controllability means that some states (or linear combination of states) cannot be arbitrarily driven to the origin, even though they may asymptotically go there. A system is observable if its state at time t can be fully reconstructed from knowing the history of the output sequence $y([0,t])$. A lack of observability means that some states (or linear combination of the states) do not influence the system output, i.e. they can take arbitrary values without that being noticeable in the output of the system.

The notions of controllability and observability are pivotal in control. Controllability ensures that one has a "full access" to the state of the system via manipulating the input of the system, such that every state can be manipulated at will by an adequate choice of input. Observability guarantees that the state of the system can be reconstructed, with is crucial for deploying state feedback techniques, and also guarantees that the state of the system can be fully monitored via the measurements (aka outputs) of the system. In an un-observable system, some state (or linear combination of states) could drift to ∞ without this being noticeable in the system outputs.

3. The following ODE describes in its simplest form the lateral dynamics of the Harrier Vertical Take-Off an Landing aircraft:

$$J\ddot{\theta} = -\xi\dot{\theta} + cu \tag{10a}$$
$$m\ddot{x} = -\xi\dot{x} + msin(\theta) \tag{10b}$$

We will use the numerical values $J = 1$, $\xi = 0.1$, and $c = 1$, $m = 1$ in the following.

(a) (2 points) Write the transfer function from the input u to the output x at steady-state.

(b) (2 points) Form a linear state-space representation of (10), from the input u to the output x at steady-state.

(c) (6 points) Form the canonical controllable form for (10) at steady-state and build a state-space controller that imposes all closed-loop poles at -1

Solution:

(a) In order to write a transfer function for (10), one needs to linearise the system first. The ODE (10) has the obvious steady-state $\theta_0 = 0$, $\dot{\theta}_0 = 0$, $u_0 = 0$ (x can take any value at steady-state here). We can then perform the linearization using $\sin(\theta_0 + \Delta\theta) \approx \Delta\theta$ and write the linear ODE:

$$J\Delta\ddot{\theta} = -\xi\Delta\dot{\theta} + c\Delta u \tag{11a}$$
$$m\Delta\ddot{x} = -\xi\Delta\dot{x} + m\Delta\theta \tag{11b}$$

The transform of this linear ODE then reads as:

$$\Delta\theta = \frac{c}{Js^2 + \xi s}\Delta u, \qquad \Delta x = \frac{m}{ms^2 + \xi s}\Delta\theta$$

such that the transfer function from Δu to Δx reads as:

$$G(s) = \frac{cm}{(ms^2 + \xi s)(Js^2 + \xi s)}$$

(b) We can form this linear state-space representation directly from (11), or apply the formal "recipe" for linearising a system. We then form first a nonlinear state-space representation of (10), it reads as:

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}}_{=X} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \frac{1}{m}\left(-\xi\dot{x} + msin(\theta)\right) \\ \frac{1}{J}\left(-\xi\dot{\theta} + cu\right) \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ -\frac{\xi}{m}\dot{x} + sin(\theta) \\ -\frac{\xi}{J}\dot{\theta} + \frac{c}{J}u \end{bmatrix}}_{=f(X,u)} \tag{12a}$$

and form a linearisation of $f(X, u)$ at $X = 0$, $u = 0$, i.e.

$$f(0 + \Delta X, 0 + \Delta u) \approx \begin{bmatrix} \Delta\dot{x} \\ \Delta\dot{\theta} \\ -\frac{\xi}{m}\Delta\dot{x} + \Delta\theta \\ -\frac{\xi}{J}\Delta\dot{\theta} + \frac{c}{J}\Delta u \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -\frac{\xi}{m} & 0 \\ 0 & 0 & 0 & -\frac{\xi}{J} \end{bmatrix}\Delta X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{c}{J} \end{bmatrix}\Delta u$$

(c) It can be hard to write the controllable canonical form from the state from the state-space form above. It is, however, quite easy to do it from the transfer function of the system computed previously. Indeed,

$$G(s) = \frac{cm}{(ms^2 + \xi s)(Js^2 + \xi s)} = \frac{1}{s^4 + 0.2s^3 + 0.01s^2}$$

such that the controllable canonical form reads as (see cheat sheet):

$$A = \begin{bmatrix} -0.2 & -0.01 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

The state feedback $K = \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \end{bmatrix}$ then produces the closed-loop dynamics $\dot{X} = (A - BK)X$, i.e.[1]:

$$\dot{X} = \begin{bmatrix} -0.2 - K_1 & -0.01 - K_2 & -K_3 & -K_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} X$$

which has the characteristic polynomial:

$$s^4 + (0.2 + K_1)s^3 + (0.01 + K_2)s^2 + K_3 s + K_4 = 0$$

Since we want all poles at -1, we want the characteristic polynomial to be of the form:

$$(s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1 = 0$$

By identification we therefore conclude that:

$$0.2 + K_1 = 4, \qquad 0.01 + K_2 = 6, \qquad K_3 = 4, \qquad K_4 = 1$$

i.e.

$$K = \begin{bmatrix} 3.8 & 5.99 & 4 & 1 \end{bmatrix}$$

[1] note that the state X vector we are using here is not equivalent to the state vector of the previous question, as the state-space form is different

4. Consider a system described by the transfer function $G(s)$ in closed-loop with a controller described by the transfer function $F(s)$. The open-loop transfer function $L(s) = F(s)G(s)$, closed-loop transfer function $T(s) = \frac{L(s)}{1+L(s)}$ and sensitivity function $S(s) = \frac{1}{1+L(s)}$ are depicted in Fig. 1.

   (a) (2 points) What is the modulus margin (i.e. the inverse of the minimum distance of the Nyquist curve to the critical point -1) ? At what frequency does it occur ?

   (b) (3 points) Does the closed-loop system have a static error ? There are three ways of answering that question, one point will be granted for each.

   (c) (2 points) What is the maximum cross-over frequency that can be achieved by adjusting (only) the gain of the controller $F(s)$?

   (d) (2 points) What are the phase and gain (amplitude) margin of the closed-loop system ?

   Answers can be found via reading the graphs. Approximate (but justified) answers are therefore accepted.

Solution:

(a) The modulus margin M_m is given by:

$$M_\mathrm{m} = \min_\omega |1 + L(j\omega)| = \max_\omega |1 + L(j\omega)|^{-1} = \max_\omega |S(j\omega)|^{-1}$$

This is information can be readily read from the Bode plot of S, giving:

$$M_\mathrm{m} = \max_\omega |S(j\omega)|^{-1} \approx -2.3\text{dB} = 0.77$$

(b) The closed-loop system does not have a static error when

$$\lim_{t\to\infty} y(t) = r, \quad \text{or equivalently, when} \quad \lim_{t\to\infty} e(t) = 0$$

Observing that:

$$E(s) = S(s)R(s), \qquad Y(s) = T(s)R(s)$$

This, in turn requires (we invoke the final value theorem here)

$$S(0) = 0, \quad \text{or equivalently} \quad T(0) = 1 \tag{13}$$

Which can be assessed in the Bode plots by looking at the magnitude plots of S and T. Alternatively, both (13) hold if $\lim_{s\to 0} |L(s)| = \infty$, which can also be assessed in the magnitude plot of L.

(c) The maximum theoretical cross-over frequency can be achieved by increasing the controller gain until $|L(j\omega)|$ crosses the 0dB axis at the frequency where $\arg[L(j\omega)] = -\pi$. From the Bode plot of L, this occurs at about 0.85 rad/s.

(d) From the Bode plot of L, the phase margin is about 110°. The gain margin is about 15dB.

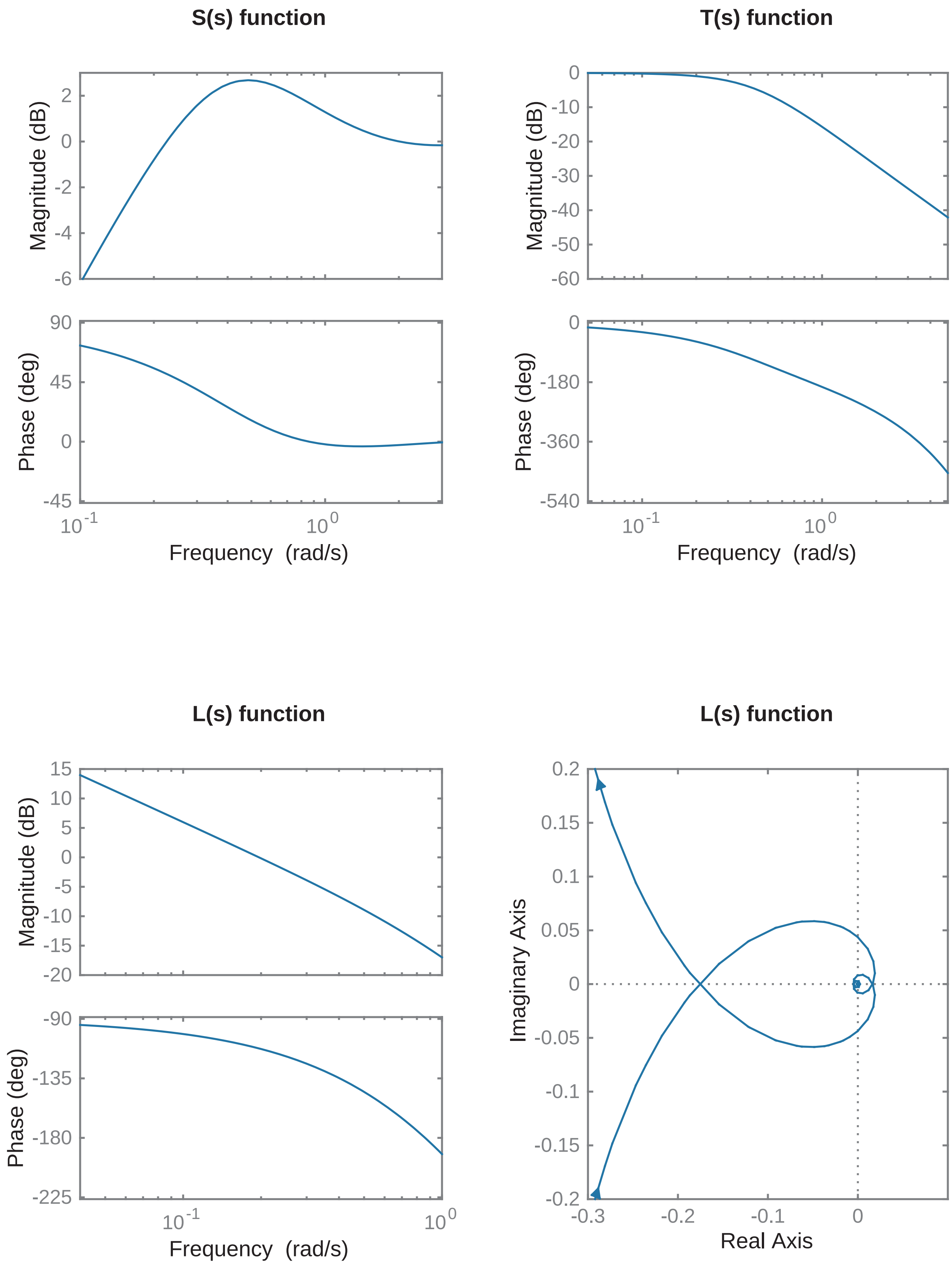

Figure 1: Nyquist and Bode plots of the transfer function $L(s)$, $S(s)$, $T(s)$.

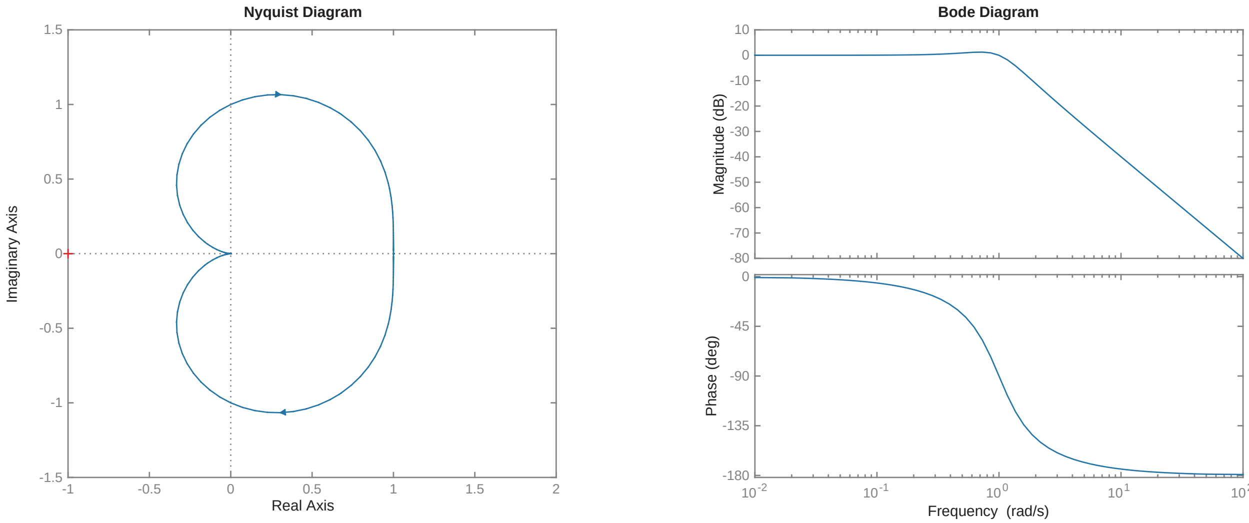

Figure 2: Nyquist and Bode plot of the transfer function $F(s)G(s)$.

5. Miscellaneous

   (a) (3 points) Consider a first-order system described by the stable transfer function $G(s)$ and a controller $F(s)$ such that the transfer function $F(s)G(s)$ is stable. The Bode and Nyquist plots of $F(s)G(s)$ are reported in Fig. 2. What is the closed-loop bandwidth that is theoretically achievable by changing the gain of controller $F(s)$? What limits that bandwidth in practice ? Explain your reasoning in both the Nyquist and Bode plot.

   (b) (3 points) Why is an unstable system more difficult to control than a stable one ? Explain thoroughly.

   (c) (2 points) Consider the stable transfer function

$$G(s) = \frac{\cosh(-2s)\left(3s^3 + 2s^2 + 1\right)}{(s+1)^4} \tag{14}$$

   and the signal $u(t)$ given in the time domain:

$$u(t) = \cos\left(e^{-0.1t}t\right) \tag{15}$$

   Compute the steady-state response of $G(s)$ subject to the input $u(t)$. Justify.

   (d) (3 points) Consider a system described by the transfer function:

$$G(s) = \frac{e^{-s}}{s+1} \tag{16}$$

   having the Nyquist diagram displayed in Fig. 3. What is the maximum cross-over frequency that can be achieved with a P-controller ? What is the corresponding controller gain ? An accurate answer will get the full grading. *Hint: you may find useful that the equation $x + \operatorname{a}\tan(x) = \pi$ is solved by $x = 2.03$.*

Solution:

(a) From both the Bode and Nyquist plots, one can see that there is no theoretical limit to the gain one can use in the controller. In the Nyquist plot, a change of gain results in a "dilatation" of the Nyquist contour. However, as far as one can observe, the Nyquist plot does not intersect

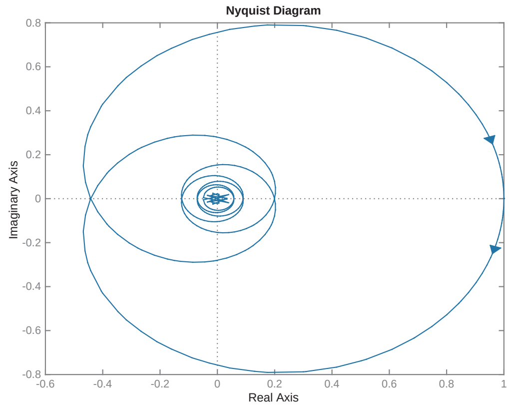

Figure 3: Nyquist and Bode plot of the transfer function $F(s)G(s)$.

the strictly negative part of the real axis, such that no dilatation can bring it to encircle the critical point -1. It follows that the system will be always stable regardless of the controller gain. A similar conclusion can be reached from observing the Bode plot. Since the phase of the open-loop transfer function never drops below -180 deg, the phase margin will always be positive regardless of the controller gain.

In practice, however, two major difficulties prevent one from increasing the controller gain indefinitely. The first is related to the robustness margin. Indeed, one can observe from both the Nyquist and the Bode plot that the robustness margins (phase margin, amplitude margin and modulus margin) decrease as the controller gain increases. Their limit value is 0, such that for very large gains there is no robustness margin left, and the slightest model error would yield an unstable closed-loop system. The second difficulty is related to the physical capabilities of the system. A larger controller gain yields more amplitude in the control signal, which would at some point lead to exceeding the limitations of the actuators and of the system itself.

(b) An unstable system is more difficult to control because of the theorem known as the Bode first integral (or Bode sensitivity integral). This theorem states that[1]:

$$\int_0^{\infty} \ln |S(i\omega)| \, \mathrm{d}\omega = \pi \sum_i \mathrm{Re}\,(p_i)$$

where $p_{1,\ldots}$ are the unstable pole of the open-loop transfer function, and Re the real-value operator. It essentially states that the surface of the sensitivity function S "around" the 0db line is conserved (in a linear-log plot because of the ln), regardless of the controller. The overall surface below and above 0db adds up to the term on the right-hand side of the equation. If the system is unstable, its unstable poles will enter in the open-loop transfer function, and therefore in the list $p_{1,\ldots}$. Unstable poles in the system therefore increase the overall surface of the sensitivity function above 0db. One needs then to remember that a high sensitivity is detrimental in terms of performance, and robustness.

[1] this expression is valid if the open-loop transfer function is of relative degree 2 or more. It takes a slightly different form if not, which does not change the proposed reasoning.

(c) Since we care only about the steady-state response, we can simply apply the final-value theorem (which is valid in this case because the system is stable). We have:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)U(s)$$

Computing the Laplace transform of $U(S)$ can be a bit tricky. However, we can use

$$\lim_{s\to 0} sU(s) = \lim_{t\to\infty} u(t) = 1 \tag{17}$$

such that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} G(s) = 1$$

(d) For a P-controller $F(s) = K$, the phase of the open-loop transfer function $F(s)G(s)$ reads as:

$$\measuredangle G(j\omega) = -\omega - \mathrm{a}\tan(\omega)$$

such that the frequency at which $F(j\omega)G(j\omega)$ gets a phase of $-180°$, labelled $\omega_{-\pi}$, is given by:

$$-\omega_{-\pi} - \mathrm{a}\tan(\omega_{-\pi}) = -\pi$$

and is $\omega_{-\pi} = 2.03$ rad/s. The cross-over frequency cannot be higher than $\omega_{-\pi}$. The controller gain K_{P} that yields $\omega_{\mathrm{c}} = \omega_{-\pi}$ is then given by:

$$|K_{\mathrm{P}} G(j\omega_{-\pi})| = 1$$

i.e.

$$K_{\mathrm{P}} |G(j\omega_{-\pi})| = K_{\mathrm{P}} \left(2.03^2 + 1\right)^{-\frac{1}{2}} = 1$$

hence $K_{\mathrm{P}} < 2.2629$ is required in order to have closed-loop stability.