ESS017 Reglerteknik E ht14
Fall 2014
Exam English
14/10/31

Student code: ______________________________

This exam contains 11 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You are allowed to use your cheat sheets and a basic calculator on this exam.

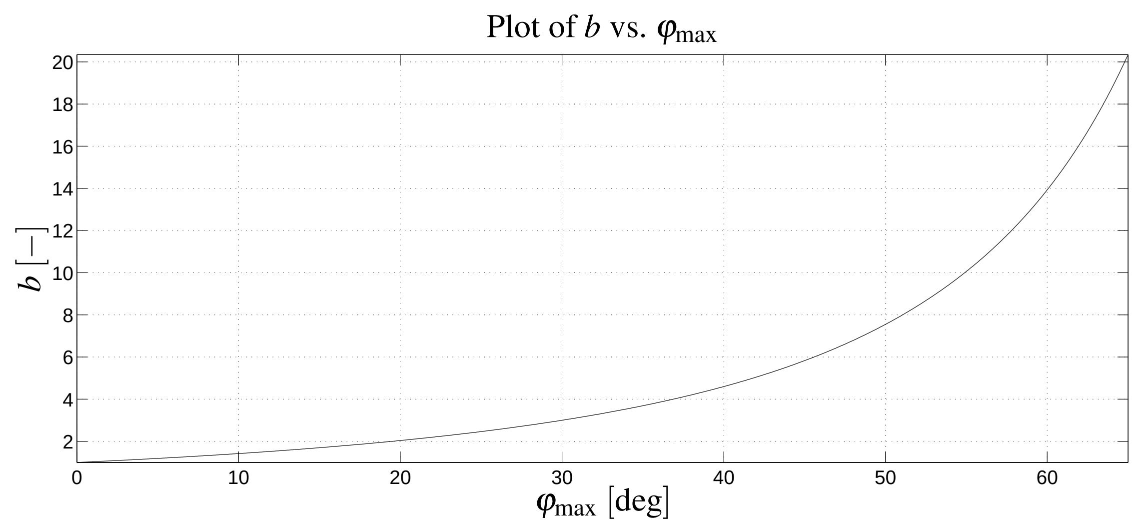
You are required to show your work on each problem on this exam. The following rules apply:

- **The maximum grade** is obtained for 33 points, hence you can fail 5 points, and still get the top grade. The passing grade will be given at 14 points.
- **Organize your work**, in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will impede the crediting.
- **Mysterious or unsupported answers** will not receive full credit. A correct answer, unsupported by clear calculations, or clear explanations will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial or even full credit.
- **Do not get lost in endless computations**. Most of the proposed questions require limited calculations if approached correctly.
- **There are hard and easier questions**, dispatched throughout the exam. The questions are not sorted by order of difficulty but by theme. If you get stuck somewhere, move on to something easier.

Do not write in the table to the right.

Problem	Points	Score
1	8	
2	7	
3	7	
4	6	
5	10	
Total:	38	

Best of luck to all !!



Figur 1: Plot of the formula $b = \frac{1+\sin\varphi_{\max}}{1-\sin\varphi_{\max}}$

1. **Controller design**:
   (a) (2 points) Consider the system $G(s) = \frac{1}{s^2}$. Sketch the Nyquist and Bode plot (you do not need much computations here, nor to make a nice plot, just a sketch !)
   (b) (2 points) Can we stabilise[1] $G(s)$ with a P controller ? Justify.
   (c) (3 points) Design a PD controller of the form
   $$F(s) = K_{\mathrm{P}}\left(\frac{1+\tau_{\mathrm{D}}s}{1+\tau_{\mathrm{D}}s/b}\right) \tag{1}$$
   so as to get a phase margin of $\phi_{\text{margin}} = 60^\circ$, and a maximum controller gain (i.e. $\max_\omega |F(j\omega)|$) of 10db. Maximize the cross-over frequency.
   (d) (1 point) Do we need an integrator in order to remove the static error ? Justify.

Solution:

(a) We can simply observe that $G(j\omega) = -\frac{1}{\omega^2}$, and has therefore a phase of -180°, and will have a gain spanning $]0,\infty[$. It is then trivial to sketch both the Bode plot and Nyquist plot.

(b) No we cannot. This can be explained in different ways, but an easy approach is the following. Let us use $F(s) = K_{\mathrm{P}}$, the closed-loop transfer function then reads as:

$$T(s) = \frac{K_{\mathrm{P}}\frac{1}{s^2}}{1+K_{\mathrm{P}}\frac{1}{s^2}} = \frac{K_{\mathrm{P}}s^2}{s^2+K_{\mathrm{P}}}$$

and takes the complex conjugate pair of poles $p_{1,2} = \pm j\sqrt{K_{\mathrm{P}}}$. These poles are on the imaginary axis for any value of K_{P} and therefore do not yield a stable time response.

(c) Since the phase of $G(s)$ is always -180°, a phase margin of $\phi_{\max} = 60^\circ$ requires a phase lift of $\phi = 60^\circ$. Using Fig. 1, this fixes constant $b = 14$. The maximum controller gain is given in the Cheat-sheet, which provides the Bode plot of $F(s)$, and reads as:

$$\max_{\omega} |F(j\omega)| = bK_{\mathrm{p}} = 10\text{db} \equiv 3.1623$$

[1] i.e. all poles strictly in the left-half-plane

i.e. $K_p = 0.226$. Finally we need to ensure that at the cross-over frequency ω_c, we have $|F(j\omega_c)G(j\omega_c)| = 1$. Here we need to place our phase-lift at ω_c in order to get the adequate phase margin. We then have

$$|F(j\omega_c)| = \sqrt{b}K_p,$$

and the cross-over frequency is provided by:

$$|F(j\omega_c)G(j\omega_c)| = \sqrt{b}K_p \frac{1}{\omega_c^2} = 1,$$

i.e. $\omega_c^2 = \sqrt{b}K_p$ which yields $\omega_c = 0.92\,\mathrm{rad/s}$. Finally, we get τ_D from the PD formula (c.f. Cheat-sheet):

$$\omega_c = \frac{\sqrt{b}}{\tau_D},$$

so that $\tau_D = \omega_c^{-1}\sqrt{b} = 4.1$

(d) No, the system $G(s)$ has already two integrators.

2. **Nyquist and Bode interpretation**: consider a stable system $G(s) = \frac{e^{-0.5s}}{(s+1)^2}$ in closed-loop with the controller $F(s) = 1 + \frac{1}{s}$, described in Figure 2 via the Nyquist and Bode plot of its open-loop transfer function $L(s) = F(s)G(s)$, the Bode plot of its sensitivity function $S(s) = \frac{1}{L(s)+1}$ and the step-response of the closed-loop transfer function $T(s) = \frac{L(s)}{L(s)+1}$ (or complementary sensitivity function).

   (a) (2 points) What is the static error of the closed-loop system ? Justify.

   (b) (1 point) What type of controller is $F(s)$? Justify.

   (c) (1 point) What Nyquist criteria applies[2] to this system ? Justify.

   (d) (2 points) What are the robustness margins (modulus, phase and amplitude) for that system ? Justify. You are allowed to provide approximate answers.

   (e) (1 point) By what ratio can the controller gain be increased before reaching instability ?

Solution:

(a) The closed-loop system has no static error. This is clear if one takes $\lim_{s\to 0} F(s)G(s) = \infty$. An infinite open-loop gain at low frequency entails that 1) the sensitivity function is zero at low frequencies (implying no residual error ensuing a reference step) and 2) the closed-loop gain is 1 at low frequency (implying that the system output follows exactly the reference at low frequency)

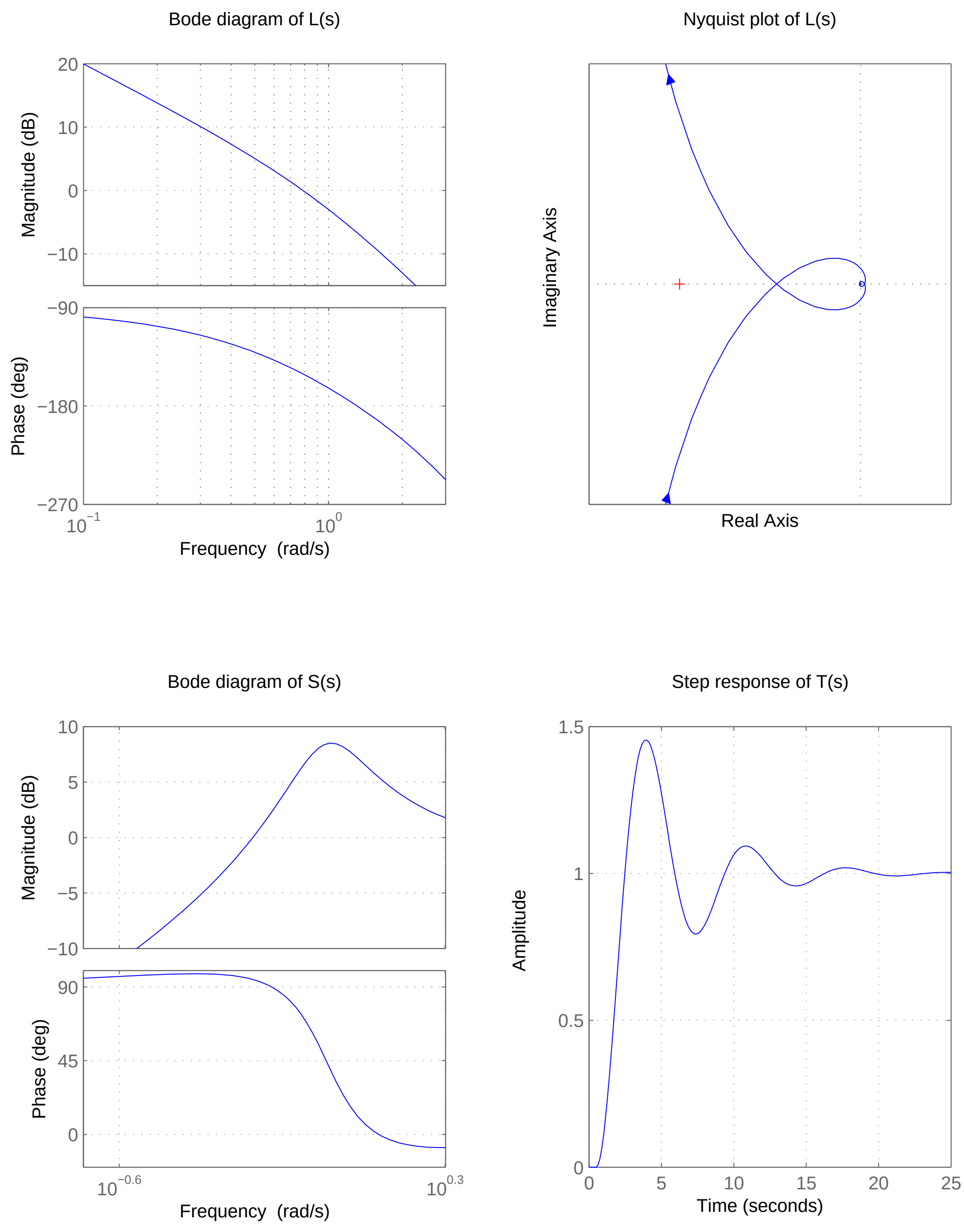
(b) This is a PI controller. The proportional gain is 1 and the integrator time constant is 1. The gain of the controller is infinite at low frequencies, and tends to 1 at high frequencies.

(c) The system is said to be stable in the description. The controller is stable (but has one pole at $s = 0$). The simplified Nyquist criteria applies.

(d) Since approximate answers are acceptable, we can read the margins directly from the plots. The phase and amplitude margins can be read in the Bode plot of $L(s)$ and are $\phi_{\text{margin}} = 29.3°$, and $A_{\text{margin}} = 6.64\text{dB}$. The modulus margin can be measured in the Bode plot of $S(s)$, by checking the top of the sensitivity function, i.e. $M_{\text{margin}} = 2.67^{-1} \equiv -8.5\text{dB}$.

(e) The Nyquist plot lacks scales, so we can not answer the question by just reading it. However, we know that when the Nyquist contour crosses the real axis, the phase of $L(s)$ is 180°, corresponds to a gain of -6.44dB (this is the robustness margin computed before). It follows that we can increase the controller gain by a factor of $6.44\text{dB} \equiv 2.1$.

[2] simplified or full



Figur 2: Bode and Nyquist plots of $L(s)$, Bode plot of $S(s)$ and step response of $T(s)$

3. **State-space**: consider the transfer function:

$$G(s) = \frac{1}{s^4} \tag{2}$$

(a) (2 points) Give a state-space representation in the observable canonical form:

$$\dot{x}(t) = \begin{bmatrix} -a_1 & 1 & \dots & 0 \\ -a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & 1 \\ -a_n & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} x(t)$$

(b) (2 points) Is this state-space representation strictly[3] stable ? Is it controllable ? Justify.

(c) (3 points) Design a state feedback controller $K \in \mathbb{R}^4$ that imposes all the closed-loop poles at -1 (i.e. the poles corresponding to the closed-loop matrix $A - BK$).

Solution:

(a) The observable canonical form reads as:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t)$$

(b) It is not strictly stable because the transfer function $G(s)$ is not strictly stable. More specifically, the eigenvalues of the "Amatrix in the observable canonical form will have four zero eigenvalues, which yield not strictly stable dynamics.

The observable canonical form arises from a transfer function that has no pole-zero cancellation and is therefore controllable. The question can also be easily verified by computing the controllability matrix associated to the observable canonical form (this can be almost without computing anything), which reads as:

$$\mathscr{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is clearly invertible and therefore full rank.

(c) To perform a pole placement we need the controllable canonical form. It reads as:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{=A} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x(t)$$

[3] all poles strictly in the left-hand-side of the complex plane

The state feedback

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} \tag{3}$$

then forms with matrix A the closed-loop matrix:

$$A - BK = \begin{bmatrix} -k_1 & -k_2 & -k_3 & -k_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

If we want all poles at -1, we need $A - BK$ to have the characteristic equation:

$$(\lambda + 1)^4 = \lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1 = 0 \tag{4}$$

By identification between the controllable canonical form and the corresponding transfer function (c.f. Cheat-sheet), we know that the first line of $A - BK$ provides the coefficients of the characteristic equation, with a change of sign, i.e. it reads as:

$$\lambda^4 + k_1\lambda^3 + k_2\lambda^2 + k_3\lambda + k_4 = 0 \tag{5}$$

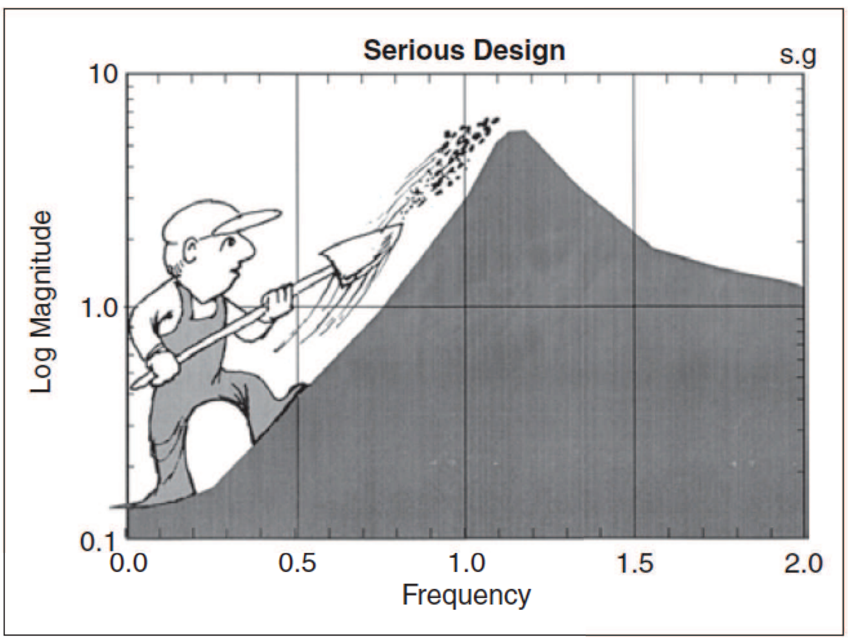
By identification between and , we see that:

$$k_1 = 4, \quad k_2 = 6, \quad k_3 = 4, \quad k_4 = 1,$$

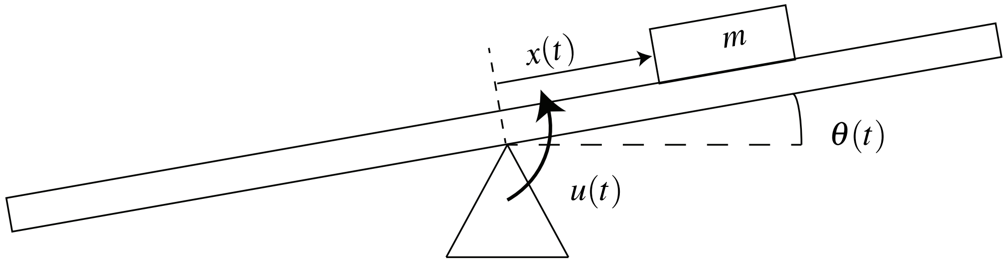
4. **Theory**: (thoroughly justified answers are expected here)
   (a) (2 points) In the Cheat-sheet you have one formula describing the limitation on the closed-loop bandwidth in relation to the model error. Report it here, explain what the different terms are, and explain its meaning.
   (b) (2 points) Discuss the fundamental trade-off in control between performance and robustness. You answer should revolve around the sensitivity function and the Bode first integral

$$\int_0^\infty \log S(i\omega)\, d\omega = 0$$

   (c) (2 points) Why are good performances harder to achieve on minimum-phase and delayed systems ? In what sense these two types of systems are related ?



Figur 3: From Respect the Unstable", G. Stein, IEEE Control System Magazine, 2003.



Figur 4: Sketch of the ball on a plate.

5. **Nonlinear system:** we consider a one-dimensional block on a plate as depicted in Fig. 4. The block can slide frictionless on the rail, and a torque can be applied to the rail to balance the block. The dynamics can be described as:

$$\ddot{x} = x\dot{\theta}^2 - g\sin(\theta) \tag{6a}$$

$$\ddot{\theta} = \frac{u - mgx\cos(\theta) + 2mx\dot{x}\dot{\theta}}{mx^2 + J} \tag{6b}$$

where $g \approx 10$ is the gravity, $m = 1$ is the mass of the block and $J = 1$ is the inertia of the rail. Variable x denotes the position of the block on the rail, θ the angle of the rail, and the input u is the torque applied at the joint of the rail.

(a) (1 point) Provide a nonlinear state-space representation of (6).

(b) (4 points) Find the steady-state point for $x = 0$ and compute a linear state-space representation approximating the system dynamics at steady-state

(c) (2 points) Is the system controllable ? Justify.

(d) (3 points) Show that the system is observable regardless of our choice for the (single) output (i.e. for x, $\dot{x}$, θ, $\dot{\theta}$).

Solution:

(a) We can e.g. pick the state vector

$$\mathrm{x}(t) = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}$$

and write the nonlinear dynamics as:

$$\dot{\mathrm{x}}(t) = \mathrm{f}(\mathrm{x}, u) = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ -x\dot{\theta}^2 - g\sin(\theta) \\ \left(mx^2 + J\right)^{-1}\left(u - mgx\cos(\theta) + 2mx\dot{x}\dot{\theta}\right) \end{bmatrix}$$

(b) The system has the trivial operating point $\mathrm{x}_0 = 0$. To proceed with the linearisation, we compute:

$$\frac{\partial}{\partial \mathrm{x}}\left(-x\dot{\theta}^2 - g\sin(\theta)\right) = \begin{bmatrix} -\dot{\theta}^2 & -g\cos(\theta) & 0 & -2x\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -10 & 0 & 0 \end{bmatrix},$$

The second linearisation is a bit more intricate, but tractable. It reads as:

$$\frac{\partial}{\partial x}\left(\left(mx^2+J\right)^{-1}\left(u-mgx\cos(\theta)+2mx\dot{x}\dot{\theta}\right)\right)=2mx\left(mx^2+J\right)^{-2}\left(u-mgx\cos(\theta)+2mx\dot{x}\dot{\theta}\right)$$
$$+\left(mx^2+J\right)^{-1}\left(-mg\cos(\theta)+2m\dot{x}\dot{\theta}\right)=\frac{mg}{J}=-10$$
$$\frac{\partial}{\partial \theta}\left(\left(mx^2+J\right)^{-1}\left(u-mgx\cos(\theta)+2mx\dot{x}\dot{\theta}\right)\right)=\left(mx^2+J\right)^{-1}mgx\sin(\theta)=0$$
$$\frac{\partial}{\partial \dot{x}}\left(\left(mx^2+J\right)^{-1}\left(u-mgx\cos(\theta)+2mx\dot{x}\dot{\theta}\right)\right)=\left(mx^2+J\right)^{-1}2mx\dot{\theta}=0$$
$$\frac{\partial}{\partial \dot{\theta}}\left(\left(mx^2+J\right)^{-1}\left(u-mgx\cos(\theta)+2mx\dot{x}\dot{\theta}\right)\right)=\left(mx^2+J\right)^{-1}(2mx\dot{x})=0$$

We finally observe that

$$\frac{\partial \mathrm{f}}{\partial u}=\begin{bmatrix}0\\0\\0\\\left(mx^2+J\right)^{-1}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

We can then compile these computations into the matrix form:

$$\Delta \mathrm{x}(t)=\underbrace{\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&-10&0&0\\-10&0&0&0\end{bmatrix}}_{=A}\Delta \mathrm{x}(t)+\underbrace{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}_{=B}\Delta u(t)$$

(c) We observe that the powers of A are trivial:

$$A^2=\begin{bmatrix}0&-10&0&0\\-10&0&0&0\\0&0&0&-10\\0&0&-10&0\end{bmatrix},\qquad A^3=\begin{bmatrix}0&0&0&-10\\0&0&-10&0\\100&0&0&0\\0&100&0&0\end{bmatrix}$$

And since the B matrix has only the last element which is not zero, the controllability matrix is easy to build, i.e. its second to last rows are the last row of A, A^2, A^3. We get:

$$\mathscr{C}=\begin{bmatrix}0&0&0&-10\\0&1&0&0\\0&0&-10&0\\1&0&0&0\end{bmatrix}$$

This matrix is clearly invertible since only one element per line/row is non-zero. The same statement can be made by observing that it can be reorganised int a diagonal matrix by simple line/row permutations. The system is therefore controllable.

(d) We can establish that by computing the four observability matrices corresponding to the four different outputs, i.e. using the output matrices:

$$C_1=\begin{bmatrix}1&0&0&0\end{bmatrix},\qquad C_2=\begin{bmatrix}0&1&0&0\end{bmatrix}$$
$$C_3=\begin{bmatrix}0&0&1&0\end{bmatrix},\qquad C_4=\begin{bmatrix}0&0&0&1\end{bmatrix}$$

It is faster, however, to observe that any output matrix C_k will pickthe k^{th} line of matrices A, A^2, A^3 to form the observability matrix. By inspection and using the same argument as in the previous question, we observe that the k^{th} line of matrices A, A^2, A^3 are systematically independent, and independent of C_k. Observability is therefore guaranteed for any choice of output.