ESS017 Reglerteknik E ht13
Winter 2014
Exam 2
15/01/14

Student code: ______________________

This exam contains 13 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You are allowed to use your cheat sheets, β handbook and calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **The maximum grade** is obtained for 30 points, hence even if you fail 10 points, you still get the top grade.
- **Organize your work**, in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will impede the crediting.
- **Mysterious or unsupported answers** will not receive full credit. A correct answer, unsupported by clear calculations, or clear explanations will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial or even full credit.
- **Do not get lost in endless computations**. Most of the proposed questions require limited calculations if approached correctly.
- **There are hard and easier questions**, dispatched throughout the exam. The questions are not sorted by order of difficulty but by theme. If you get stuck somewhere, move on to something easier.

Do not write in the table to the right.

Problem	Points	Score
1	8	
2	8	
3	6	
4	8	
5	10	
Total:	40	

Best of luck to all !!

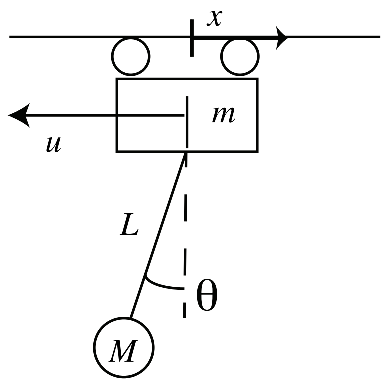

Figure 1: Sketch of the crane.

1. **Controller design**: consider the overhanging crane sketched in Fig. 1, which consists in a chariot moving on a rail (position x), controlling a mass hanging below (angle θ). The force on the chariot u is the system input. We wish to control the horizontal position of the hanging mass, given by $x_{\text{mass}} = x + L\sin\theta$. The nonlinear model of the crane has been linearised at the reference $x = 0$, $\dot{x} = 0$, $\theta = 0$, $\dot{\theta} = 0$ yielding the transfer function depicted in Figure 2.

   (a) (2 points) What type of controller should be used a) in the absence of disturbance, and b) if external disturbances are present ? Justify your answer.

   (b) (4 points) Suppose that a PD controller in the *lead* form:

   $$F(s) = K_{\text{P}}\left(\frac{1+\tau_{\text{D}}s}{1+\tau_{\text{D}}s/b}\right) \tag{1}$$

   is selected. Propose a design K_{P}, τ_{D}, b minimising the static error while retaining a phase margin of about 45 [deg]. The lead filter shall not exceed a phase advance of 60 [deg]. The Bode plot of the system transfer function $G(s)$ is displayed in Figure 2.

   (c) (2 points) A PD with a phase advance of $\varphi_{\text{max}} = 85$ [deg] occurring at $\omega = 0.75$ [rad/s] has been designed, yielding the open-loop transfer function $L(s)$ depicted in Figure 3, which has a phase margin of $\varphi_{\text{margin}} = 45$ [deg]. Comment on this design, justify your answer. (*hint: look at the amplitude margin*).

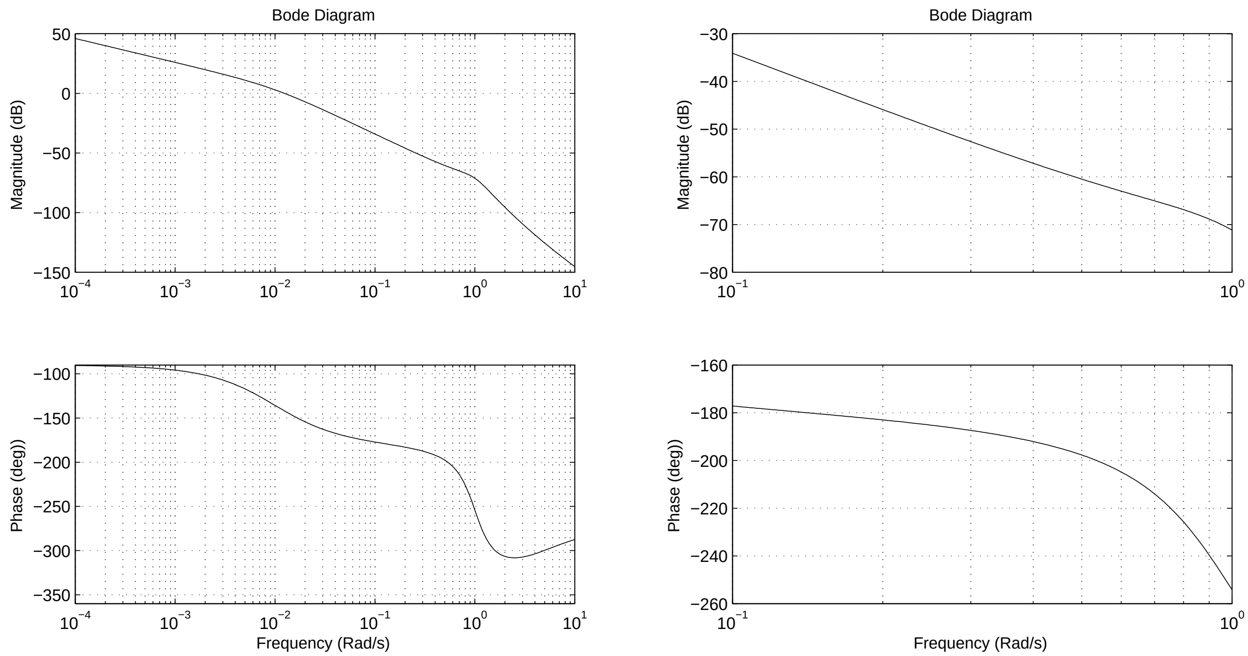

Figure 2: Bode plot of the transfer function $G(s)$ for the crane (left-hand graphs), and zoom on the frequency range $[0.1, 1]$ [rad/s] (right-hand graphs).

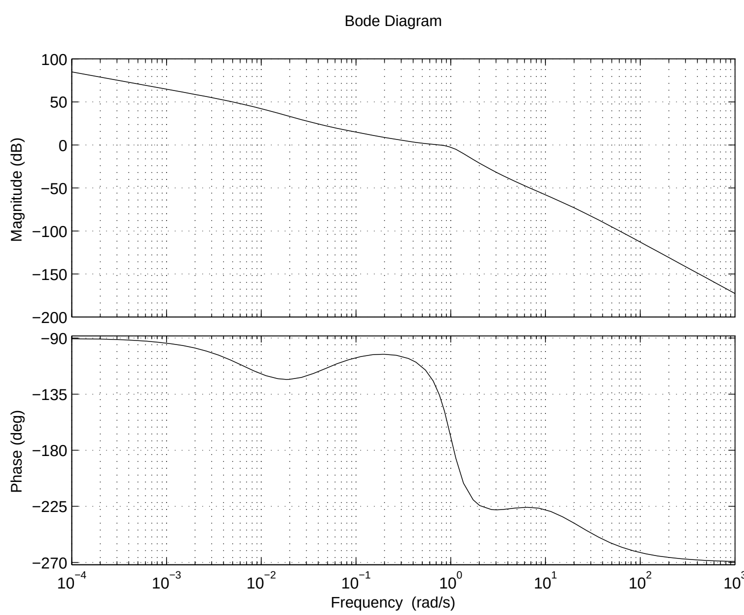

Figure 3: Bode plot of the open-loop transfer function $L(s)$ for a phase margin of $45\,[\text{deg}]$ and a phase advance of $85\,[\text{deg}]$.

Solution:

(a) In can be seen from the bode plot of the crane transfer function that the gain at low frequencies tends to infinity (slope of $-20\,[\mathrm{db/dec}]$ at low frequencies). It follows that the open-loop transfer function $L(s) = F(s)G(s)$ tends to infinity for $s \to 0$, hence an integrator is not needed in the absence of disturbance. In the presence of disturbances, an integrator is needed. Since the phase of $G(s)$ is falling below $-180\,[\mathrm{deg}]$ a D (or lead filter) is required to correct the phase and improve the closed-loop performance.

(b) One should use the largest possible b. A phase advance of $\varphi_{\mathrm{max}} = 60\,[\mathrm{deg}]$ is allowed, which corresponds to a value of $b \approx 14$ (see formula for b in the cheat-sheet). The maximal phase advance of the lead filter is best placed at the cross-over frequency, i.e. $\angle F(j\omega_{\mathrm{c}}) = \varphi_{\mathrm{max}} = 60\,[\mathrm{deg}]$. Using the formula (c.f. cheat-sheets):

$$\angle F(j\omega_{\mathrm{c}}) = -180 + \varphi_{\mathrm{margin}} - \angle G(j\omega_{\mathrm{c}})$$

with $\varphi_{\mathrm{max}} = 60\,[\mathrm{deg}]$ and the required phase margin of $\varphi_{\mathrm{margin}} = 45\,[\mathrm{deg}]$, we find:

$$\angle G(j\omega_{\mathrm{c}}) = -180 + 45 - 60 = -195\,[\mathrm{deg}]$$

We find by inspection in the bode plot that $\omega_{\mathrm{c}} \approx 0.45\,[\mathrm{rad/s}]$ fullfills this condition with (approximate answers are accepted here):

$$|G(j\omega_{\mathrm{c}})| = -59\,[\mathrm{db}]$$

With $\omega_{\mathrm{c}} = 0.45\,[\mathrm{rad/s}]$, we can find τ_{D} so as to set the frequency of the maximum phase advance of the controller $F(s)$, using $\omega_{\mathrm{c}} = \frac{\sqrt{b}}{\tau_{\mathrm{D}}}$ we get:

$$0.45 = \frac{3.74}{\tau_{\mathrm{D}}} \quad \Rightarrow \quad \tau_{\mathrm{D}} = 8.2$$

Finally, we compute K_{P} using $|G(j\omega_{\mathrm{c}})| = -59\,[\mathrm{db}]$. Indeed, at ω_{c} we have:

$$|L(j\omega_{\mathrm{c}})| = |F(j\omega_{\mathrm{c}})|\,|G(j\omega_{\mathrm{c}})| = 1 \equiv 0\,[\mathrm{db}].$$

It follows that $|F(j\omega_{\mathrm{c}})| = |G(j\omega_{\mathrm{c}})|^{-1} = 59\,[\mathrm{db}] \equiv 891$. We finally use (see the cheat-sheets):

$$|F(j\omega_{\mathrm{c}})| = \sqrt{b}K_{\mathrm{P}} = 891 \quad \Rightarrow \quad K_{\mathrm{P}} = 238$$

(c) Even though the phase margin is reasonably good, the amplitude (or gain) margin is $|L(j\omega_{\mathrm{m}})|^{-1} = 1.6$ with $\omega_{\mathrm{c}} = 1.08\,[\mathrm{rad/s}]$ given by $\angle L(j\omega_{\mathrm{m}}) = -180\,[\mathrm{deg}]$. This low amplitude margin results from a phase dropping quickly in the transfer function $L(s)$ after the phase advance provided by the PD controller, with an amplitude remaining high (see the frequency range $[0.3, 1]$ in Figure 3).

2. **Sensitivity & Nyquist**: an engineer has developed a controller of the form $F(s) = K \cdot F_0(s)$, where $F_0(s)$ is a stable transfer function and $K \in \mathbb{R}$, for a system having the model $G(s)$. Figure 4 charts the Nyquist diagram of the open-loop transfer function $L(s) = F(s)G(s)$ (upper-left). The minimal distance of the Nyquist curve to the critical point -1 in the complex plane is $0.27\,[\mathrm{s}^{-1}]$ (reported in the graph). The poles ($\times$) and zeros ($\circ$) of $L(s)$ are reported in the lower-left graph. The Bode plot of the open loop transfer function $L(s)$ is provided in Figure 4, upper-right, and the Bode plot of the sensitivity function $S(s) = \frac{1}{1+L(s)}$ is provided in Figure 4, lower-right. Some specific values of $S(j\omega)$ and $L(j\omega)$ are provided in the following tables.

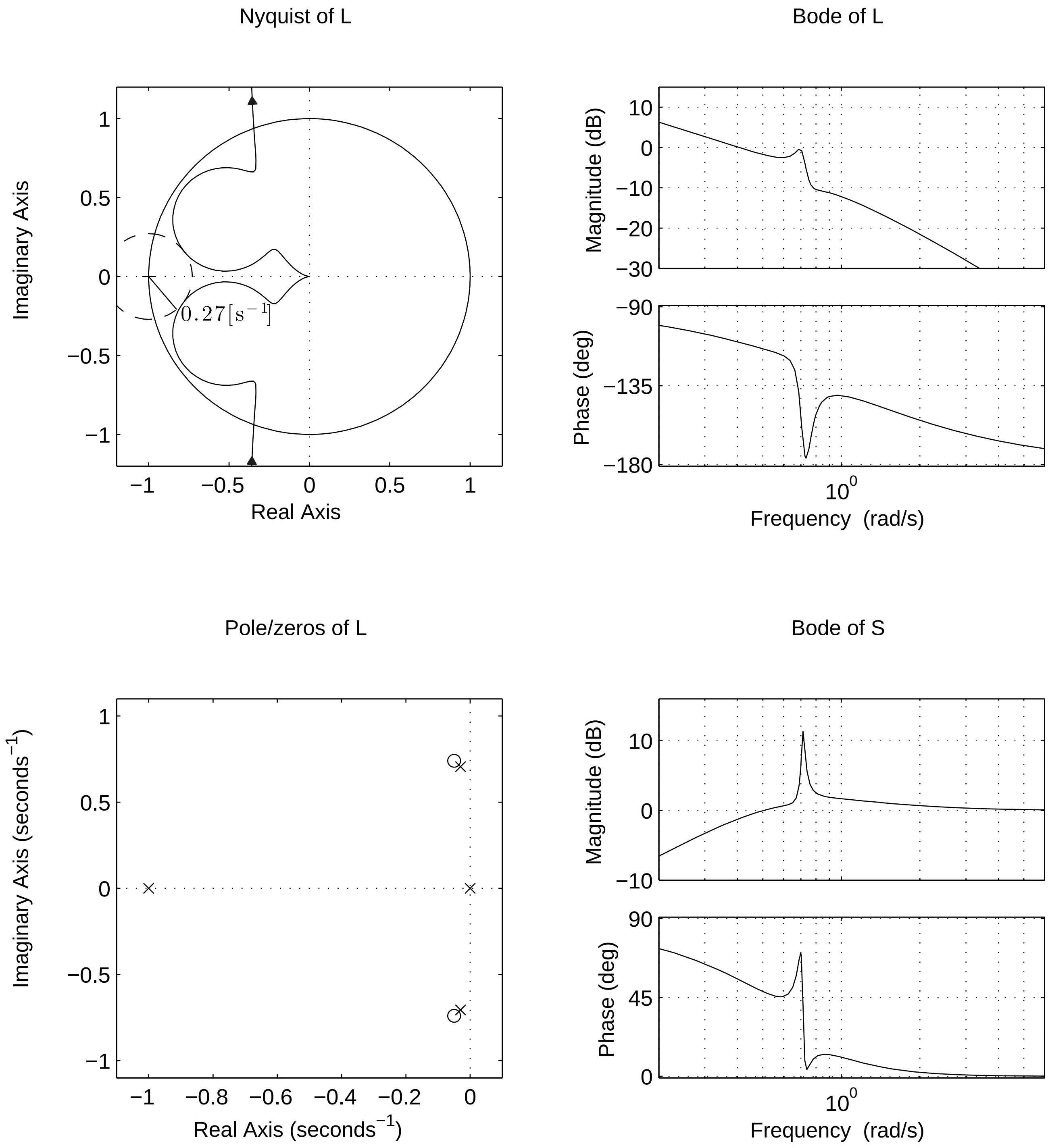

Figure 4: Poles and zeros of the closed-loop $T(s)$, of the open loop transfer function $L(s)$, Bode plot and Nyquist plot of the open loop transfer function $L(s)$.

Tab 1: Specific values of the Bode plot of $L(s)$

Frequency (rad/s)	Magnitude (db)	Phase (deg)
0	∞	-90
0.41	0	-110
∞	$-\infty$	-180

Tab 2: Specific values of the Bode plot of $S(s)$

Frequency (rad/s)	Magnitude (db)	Phase (deg)
0	$-\infty$	90
0.71	11.36	41
∞	0	0

(a) (2 points) Compute the phase margin of the closed-loop system, provide an exact number. Justify.

(b) (2 points) Compute the modulus margin of the closed-loop system, provide an exact number, justify. There are two ways of doing so from the information provided, explain both.

(c) (2 points) What is the minimum and maximum value of the controller gain K that can be used without loosing the closed-loop stability ? Justify.

(d) (2 points) The controller has been designed with a very good phase margin. Would you qualify it as a good design provided that the model of the system is far from perfect ? Explain why.

Solution:

(a) The phase margin is provided by the phase of the open-loop transfer function $L(s)$ at the cross-over frequency ω_c given by $|L(\omega_c)| = 0$. This value is provided in Table 1, i.e. $\arg(L(\omega_c)) = -110\,[\text{deg}]$. The resulting phase margin is $-180 - (-110) = 70\,[\text{deg}]$.

(b) We observe in the Nyquist Figure 4 (upper-left) that the smallest distance d_{min} of the Nyquist curve to the critical point -1 is $d_{\text{min}} = 0.27 \equiv -11.36\,[\text{db}]$. The modulus margin is given by $d_{\text{min}} = -11.36\,[\text{db}]$. The same result can be reached by finding the maximum module of the sensitivity function, i.e. $S_{\text{m}} = \max_\omega |S(j\omega)| = 11.36\,[\text{db}]$ provided in Table 2, and visible in the Bode plot of $S(s)$ Figure 4 (lower-right). The modulus margin is then provided by S_{m}^{-1}.

(c) The Nyquist curve does not intersect the real axis (though it tends asymptotically to the point $(0,0)$ for $\omega \to 0$). Therefore, no "scaling" of the Nyquist curve can enclose the critical point -1. As a result, the controller gain K does not have an upper or lower bound, even though the modulus margin can be extremely small if K is doubled. A similar conclusion could be reached by observing that the open-loop transfer function does not crosses over $-180\,[\text{deg}]$ (visible on the Bode plot of $L(s)$ and observing that the relative degree is 2, c.f. the pole-zero map), yielding an infinite amplitude margin.

(d) It is not a good design. The phase margin is excellent, but the modulus margin is poor and brings the Nyquist curve close to the critical point -1. Applying the controller to a poorly modelled system may result in closed-loop instability. This illustrates the fact that the phase margin can be a poor metric to assess the closed-loop robustness.

3. **State-space**: consider the transfer function:

$$G(s) = \frac{s^2+s+1}{2s^3+2s^2+s+1} \tag{2}$$

(a) (2 points) Give a state-space representation in the controllable canonical form. Justify.

(b) (2 points) Is the state-space representation a minimal realisation ? Is it observable ? Justify.

(c) (2 points) Design a state feedback that imposes all closed-loop poles at -1.
Hints: remember that the poles of close-loop state-space representation $\dot{x} = (A - BK)x$ *in the controllable canonical form are given by the characteristic equation* $\lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0$ *where* $\begin{bmatrix} -\alpha_1 & -\alpha_2 & \ldots & -\alpha_n \end{bmatrix}$ *is the first line of* $A - BK$.

Solution:

(a) The coefficient of the highest power of the numerator must be unitary, hence we start with rewriting $G(s)$ as:

$$G(s) = \frac{\frac{1}{2}s^2 + \frac{1}{2}s + \frac{1}{2}}{s^3 + s^2 + \frac{1}{2}s + \frac{1}{2}}$$

and then apply the formula for the controllable canonical form (see cheat sheets, and "Reglerteknik" p. 158), which yields:

$$\dot{x} = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

(b) Yes to both: canonical forms (controllable or observable) are always minimal representations and both controllable and observable. Both questions can also be answered via direct computations.

(c) Applying the state feedback $u = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} x$ to the system in its controllable canonical form yields the closed-loop dynamics (see hints):

$$\dot{x} = \begin{bmatrix} -(1+k_1) & -(\frac{1}{2}+k_2) & -(\frac{1}{2}+k_3) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$$

which has the characteristic equation:

$$\lambda^3 + (1+k_1)\lambda^2 + (\frac{1}{2}+k_2)\lambda + (\frac{1}{2}+k_3) = 0 \tag{3}$$

We want all solutions of (3) to be at $\lambda = -1$, i.e. we want (3) to be:

$$(\lambda+1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \tag{4}$$

By identification between (3) and (4), we get:

$$1 + k_1 = 3, \qquad \frac{1}{2} + k_2 = 3, \qquad \frac{1}{2} + k_3 = 1$$

i.e.

$$k_1 = 2, \qquad k_2 = \frac{5}{2}, \qquad k_3 = \frac{1}{2}$$

4. **Theory**: (thoroughly justified answers are expected here)

   (a) (2 points) Whenever the open-loop transfer function $L(s)$ has a relative degree higher or equal to 2, the Bode first integral states that:

$$\int_0^\infty \log |S(j\omega)| d\omega = \sum_{k=1}^{N} \mathrm{Re}\{p_k\} \tag{5}$$

   where $p_1, ..., p_N$ is the collection of unstable poles of the open-loop transfer function $L(s)$, and $S = \frac{1}{1+L}$ is the sensitivity function of the closed-loop system. Explain the consequences of (5) for control. Moreover, in the light of (5), explain why and in what sense an unstable system is generally harder to control than a stable one. Use illustrative sketches to support your explanations.

   (b) (2 points) A delayed system is harder to control than a non-delayed one. The longer the delay, the worse the problem is. Explain thoroughly why. Use illustrative sketches to support your explanations.

   (c) (2 points) Explain the difference between the simplified and the full Nyquist criteria. When does the simplified apply ? How does one apply the full Nyquist criteria ?

   (d) (2 points) Explain in detail the meaning and consequences of the inequality (c.f. cheat-sheets):

$$|T(j\omega)| \leq \frac{1}{|\Delta G(j\omega)|}, \quad \forall \omega \geq 0$$

Solution:

(a) The Bode first integral is a statement on the balance between the area below 0db and above 0db in the log-linear plot of the magnitude of the sensitivity function $|S(j\omega)|$. In order to increase the closed-loop performance one needs to increase (or "dig") the area below 0db, and as a consequence increase the area above (Fig. 5 is an excellent illustration of that concept).

Generally, in control, we dislike large sensitivities. When the sensitivity is large, a) model errors have a strong impact on the closed-loop transfer function, b) the modulus margin, and therefore the robustness is poor.

For a stable system, the area above 0db matches the area below 0db. For an unstable system, however, the area above 0db in excess of the area below, to compensate for the unstable poles. This often increases dramatically the sensitivity above 0db and makes the controller harder to design.

(b) Non-minimum phase systems are harder to control, in the sense that the compromise performance-robustness is more critical than for minimum-phase systems. Non-minimum phase systems have poles and/or zeros in the right-half plane (RHP), which result in an early and strong drop in the phase of the transfer function $G(s)$ of the system. The phase of $G(s)$ then crosses -180 deg early, and steeply, which can often not be pushed up effectively. Getting a reasonable phase margin then requires having a rather low cross-over frequency, hence a low bandwidth, and therefore relatively low control performances. The phase drop increases with the delay, hence a longer delay makes the problem worse.

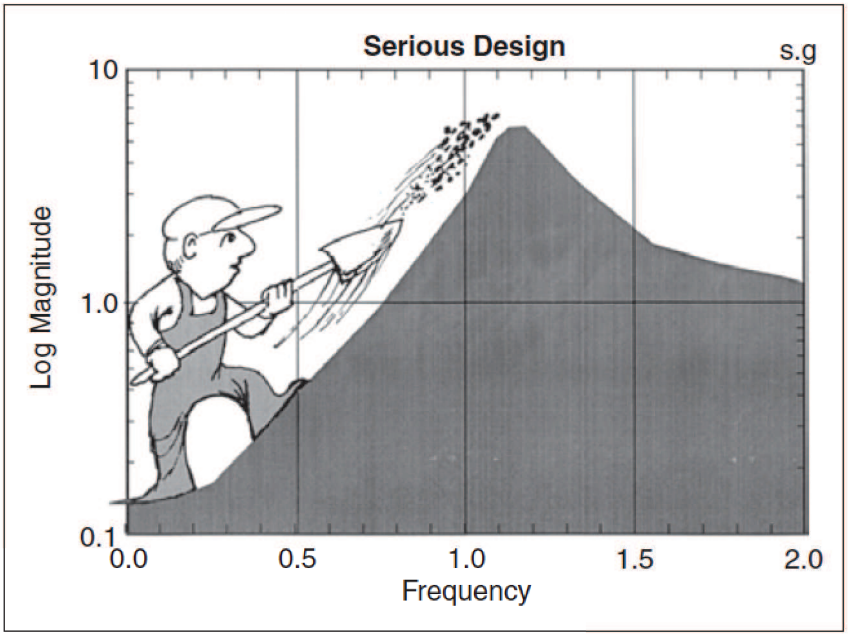

Figure 5: From "Respect the Unstable", G. Stein, IEEE Control System Magazine, 2003.

(c) The simplified Nyquist criteria applies when the open-loop transfer function $L(s)$ is stable. In that case, it is sufficient to check that the Nyquist curve does not encircle the critical point -1 in the complex plane. If the open-loop transfer function $L(s)$ is unstable, then the simplified criteria does not apply, and the full criteria must be used. The full criteria requires that the Nyquist curve encircles the critical point -1 as many times as $L(s)$ has unstable poles. Note that the direction (clockwise or counterclockwise) in which the Nyquist curve encircles -1 matters, and depends on which direction we span the imaginary axis when constructing the Nyquist contour.

(d) The formula $|T(j\omega)| \leq |\Delta G(j\omega)|^{-1}, \ \forall \omega \geq 0$ establishes an upper bound to the amplitude of the closed-loop transfer function, i.e. $|T(j\omega)|$ based on the amplitude of the relative model error:

$$\Delta G(s) = \frac{G_0(s) - G(s)}{G(s)},$$

in order to keep closed-loop stability. It essentially states that at frequencies ω where the relative model error $|\Delta G(j\omega)|$ is large, then the amplitude of the closed-loop transfer function, i.e. $|T(j\omega)|$, must be small, otherwise stability can be lost.

High closed-loop performance requires that the closed-loop transfer function $|T(j\omega)| \approx 1$ up to high frequencies. Therefore, the proposed inequality imposes a limit to the closed-loop performance, in the sense that whenever $|\Delta G(j\omega)| > 1$, then $|T(j\omega)|$ cannot be close to 1, and the closed-loop performance is degraded. Note that the relative model error is typically high at higher frequencies, where the model gain is small, and where un-modelled dynamics are often present.

5. **Nonlinear system**

A satellite orbiting the earth is described by the equations:

$$\ddot{r} = r\dot{\theta}^2 - \frac{GM}{r^2}, \tag{6}$$

$$\ddot{\theta} = \frac{T}{mr} - \frac{2}{r}\dot{r}\dot{\theta}. \tag{7}$$

where (r, θ) is the position of the satellite in polar coordinates form the center of the earth (see Fig. 6), and T is the thrust tangential to the satellite trajectory.

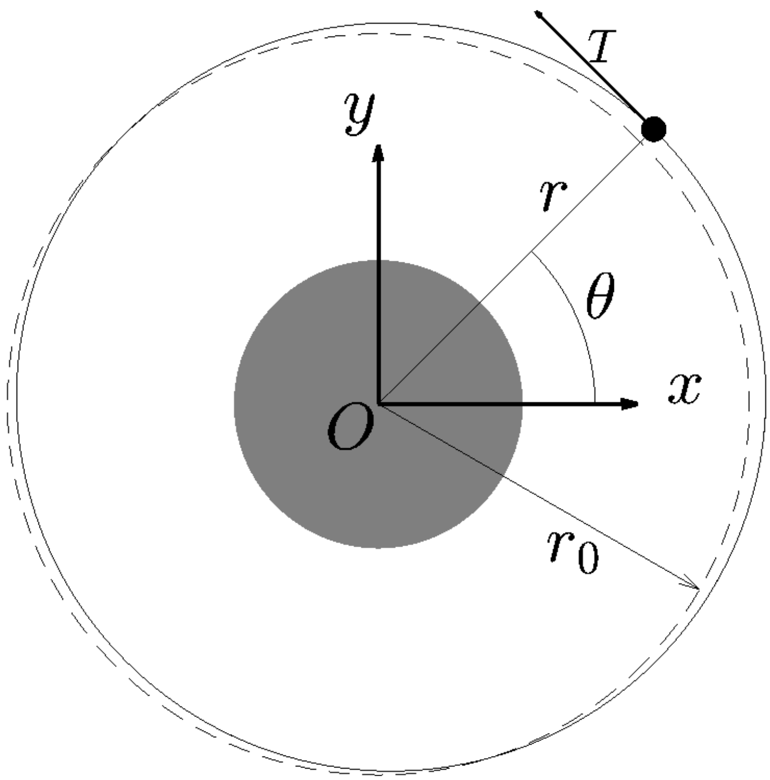

Figure 6: Schematic of the satellite problem.

We consider the problem of stabilising the orbit altitude r at the steady-state value $r = r_0$ by adjusting the thrust T. To that end we want to study the input-output relationship $T \rightarrow r$:

(a) (2 points) Provide a nonlinear state-space representation of (6)-(7) for the output r. *Hint: being careful about how many states you need to answer that question will save you some work in the following.*

(b) (2 points) Find the steady-state point corresponding to a given orbit r_0 *Hint: the angular velocity* $\dot{\theta} \neq 0$ *at steady-state, therefore* θ *does not have a steady-state value*

(c) (2 points) Compute the corresponding linearised state-space representation from input T to the output r.

(d) (2 points) Compute the controllability matrix.

(e) (2 points) Is the system controllable ? Justify.

Solution:

(a) A possible nonlinear state-space representation reads:

$$x = \begin{bmatrix} r & \dot{r} & \dot{\theta} \end{bmatrix}^T, \qquad y = r$$
$$\dot{x} = f\left(r, \dot{r}, \dot{\theta}, u\right) = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 - \frac{GM}{r^2} \\ \frac{u}{mr} - \frac{2}{r}\dot{r}\dot{\theta} \end{bmatrix} \tag{8}$$

with the input $T = u$. Note that θ is not needed to describe the dynamic $u \to r$, and can be omitted as a state. If included anyway, the state-space representation reads:

$$x = \begin{bmatrix} r & \theta & \dot{r} & \dot{\theta} \end{bmatrix}^T, \qquad y = r$$
$$\dot{x} = f\left(r, \dot{r}, \dot{\theta}, u\right) = \begin{bmatrix} \dot{r} \\ \dot{\theta} \\ r\dot{\theta}^2 - \frac{GM}{r^2} \\ \frac{u}{mr} - \frac{2}{r}\dot{r}\dot{\theta} \end{bmatrix} \tag{9}$$

Such a state-space representation is, however, not observable (θ cannot be estimated via the evolution of the output r), and therefore not minimal.

(b) The steady-state point can be computed from r_0 using:

$$r = r_0, \quad \dot{r} = 0$$
$$r\dot{\theta}^2 - \frac{GM}{r^2} = 0$$
$$\frac{T}{mr} - \frac{2}{r}\dot{r}\dot{\theta} = 0$$

which yields the steady-state point:

$$r = r_0, \qquad \dot{r} = 0, \qquad \dot{\theta} = \left(\frac{GM}{r_0^3}\right)^{\frac{1}{2}}, \qquad T = 0. \tag{10}$$

(c) We linearise the dynamics (8) at the previously computed steady-state point:

$$\Delta\dot{x} = \underbrace{\frac{\partial f}{\partial x}}_{\equiv A} \Delta x + \underbrace{\frac{\partial f}{\partial u}}_{\equiv B} \Delta u$$
$$\Delta r = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\equiv C} \Delta x$$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial u}$ are evaluated at the point (10). This yields:

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ \dot{\theta}^2 + \frac{2GM}{r^3} & 0 & 2r\dot{\theta} \\ \frac{2m\dot{r}\dot{\theta} - T}{mr^2} & -\frac{2\dot{\theta}}{r} & -\frac{2\dot{r}}{r} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{3GM}{r_0^3} & 0 & 2\left(\frac{GM}{r_0}\right)^{\frac{1}{2}} \\ 0 & -2\left(\frac{GM}{r_0^5}\right)^{\frac{1}{2}} & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{mr_0} \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

If using the non-minimal representation (9) the linearisation read as:

$$\tilde{A} = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dot{\theta}^2 + \frac{2GM}{r^3} & 0 & 0 & 2r\dot{\theta} \\ \frac{2m\dot{r}\dot{\theta}-T}{mr^2} & 0 & -\frac{2\dot{\theta}}{r} & -\frac{2\dot{r}}{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3GM}{r_0^3} & 0 & 0 & 2\left(\frac{GM}{r_0}\right)^{\frac{1}{2}} \\ 0 & 0 & -2\left(\frac{GM}{r_0^5}\right)^{\frac{1}{2}} & 0 \end{bmatrix}$$

$$\tilde{B} = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{mr_0} \end{bmatrix}, \qquad \tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

(d) Using the minimal representation (8), the controllability matrix reads:

$$\mathscr{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \frac{1}{mr_0} \begin{bmatrix} 0 & 0 & 2\left(\frac{GM}{r_0}\right)^{\frac{1}{2}} \\ 0 & 2\left(\frac{GM}{r_0}\right)^{\frac{1}{2}} & 0 \\ 1 & 0 & -\frac{4GM}{r_0^3} \end{bmatrix}. \tag{11}$$

By inspection, we see that the columns of $\mathscr{C}$ are independent, hence the state-space representation is controllable.

If the non-minimal representation (9) is used, the computation of the controllability matrix takes more effort, but is manageable, and reads as:

$$\mathscr{C} = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} & \tilde{A}^3\tilde{B} \end{bmatrix} = \frac{1}{mr_0} \begin{bmatrix} 0 & 0 & 2\left(\frac{GM}{r_0}\right)^{\frac{1}{2}} & 0 \\ 0 & 1 & 0 & -\frac{4GM}{r_0^3} \\ 0 & 2\left(\frac{GM}{r_0}\right)^{\frac{1}{2}} & 0 & -\frac{2GM}{r_0^2}\left(\frac{GM}{r_0^3}\right)^{\frac{1}{2}} \\ 1 & 0 & -\frac{4GM}{r_0^3} & 0 \end{bmatrix} \tag{12}$$

In order to assess the controllability of the linearised system, we need to check the determinant or rank of $\mathscr{C}$ (equivalent since $\mathscr{C}$ is square). To avoid taking the determinant of a 4×4 matrix, we observe that columns 1 and 3 of $\mathscr{C}$ are linearly independent, and independent of columns 2 and 4. It follows that $\mathscr{C}$ is full rank if columns 2 and 4 are independent, i.e. if and only if:

$$\mathrm{Det}\left(\begin{bmatrix} 1 & -\frac{4GM}{r_0^3} \\ 2\left(\frac{GM}{r_0}\right)^{\frac{1}{2}} & -\frac{2GM}{r_0^2}\left(\frac{GM}{r_0^3}\right)^{\frac{1}{2}} \end{bmatrix}\right) = \frac{6GM\left(\frac{GM}{r_0^3}\right)^{\frac{1}{2}}}{r_0^2} \neq 0$$

which clearly holds. The system is therefore controllable.