ESS017 Reglerteknik E ht13
Fall 2013
Exam 1
25/10/13
Time Limit: ... Minutes

Name (Print): ______________________________

This exam contains 13 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are allowed to use your cheat sheets, β handbook and calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **The maximum grade** is obtained for 30 points, hence even if you fail 10 points, you still get the top grade.
- **Organize your work**, in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will impede the crediting.
- **Mysterious or unsupported answers** will not receive full credit. A correct answer, unsupported by clear calculations, or clear explanations will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial or even full credit.
- **Do not get lost in endless computations**. Most of the proposed questions require limited calculations if approached correctly.
- **There are hard and easier questions**, dispatched throughout the exam. The questions are not sorted by order of difficulty but by theme. If you get stuck somewhere, move on to something easier.

Do not write in the table to the right.

Problem	Points	Score
1	8	
2	6	
3	6	
4	6	
5	14	
Total:	40	

Best of luck to all !!

Table 1: Model parameters

Parameter	Description	Value
m	Overall inertia	0.1 [kg]
ξ	Overall friction	$0.1\,[\mathrm{N}\cdot\mathrm{s}\cdot\mathrm{m}^{-1}]$
K_M	Motor gain	$11\,[\mathrm{N}\cdot\mathrm{V}^{-1}]$
τ	Motor time constant	0.01 [s]

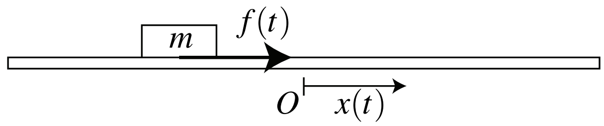


Figure 1: Sketch of the positioning system.

1. **Controller design**: consider the system sketched in Fig. 1, which consists in a mass moving on a rail, controlled by a high-power linear motor. This kind of system is used in many industrial applications as a high-speed, high-accuracy positioning system. We use the following first-principle model:

$$m\ddot{x}(t) = -\xi\dot{x}(t) + f(t) \tag{1}$$

to describe the system dynamics, where $x(t)$ is the mass position, $f(t)$ the force developed by the linear motor, and $d(t)$ an unknown load. The electrical dynamics of the motor are given by the following first-order transfer function:

$$F(s) = \frac{K_\mathrm{E}}{\tau s + 1} U(s) \tag{2}$$

where u is the voltage applied to the motor. The numerical values of the parameters can be found in Table 1.

(a) (2 points) Compute the transfer function $G(s)$ from input $U(s)$ to the output $X(s)$, i.e. $X(s) = G(s)U(s)$.

(b) (2 points) What is the simplest controller[1] needed to perform *accurate* (i.e. no static error) and *fast* (i.e. maximising the closed-loop bandwidth) position control on the system in the absence of disturbance ? Justify your answer.

(c) (4 points) Suppose that a PD controller in the *lead* form:

$$F(s) = K_\mathrm{P}\left(\frac{1+\tau_\mathrm{D}s}{1+\tau_\mathrm{D}s/b}\right) \tag{3}$$

is selected. Propose a design K_P, τ_D, b maximising the closed-loop bandwidth while retaining a phase margin of about 60 [deg]. It is desired that

$$\frac{\max_\omega |F(j\omega)|}{\min_\omega |F(j\omega)|} \leq 23\,\mathrm{db}.$$

The Bode plot of the system transfer function $G(s)$ is displayed in Figure 2. The relationship b vs. the phase lift $\varphi_{\max}$ of the PD controller (3) is charted in Figure 3. *Hint: start with computing b, then the phase that you need in $G(j\omega)$ at the cross-over frequency. This will give you the rest.*

[1] P, PD, or PID

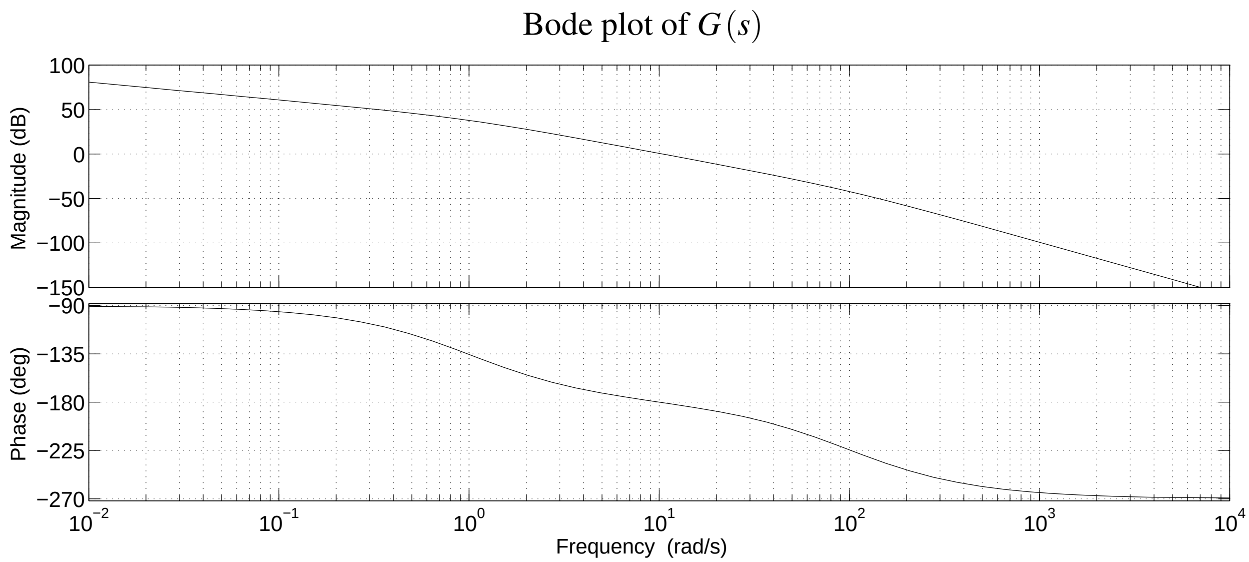


Figure 2: Bode plot of the transfer function $G(s)$ for the positioning system.

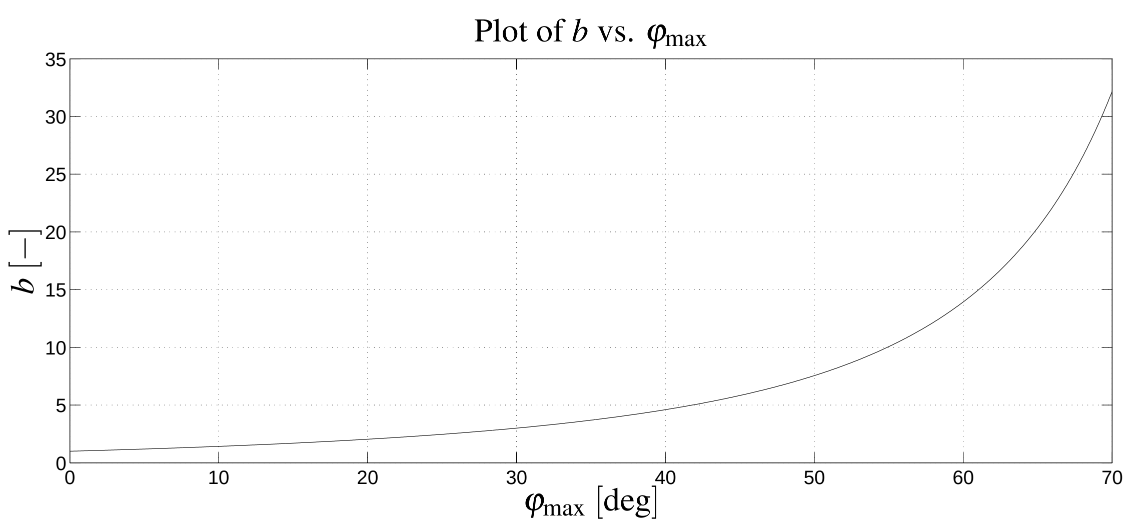


Figure 3: Plot of the formula $b = \frac{1+\sin\varphi_{\text{max}}}{1-\sin\varphi_{\text{max}}}$

Solution:

(a) The transfer function $F(s)$ to $X(s)$ reads

$$G_F(s) = \frac{1}{s(ms+\xi)}, \qquad X(s) = G_F(s)U(s)$$

hence the transfer function $G(s)$ from $U(s)$ to $X(s)$ reads:

$$X(s) = \frac{K_{\mathrm{E}}}{s(ms+\xi)(\tau s+1)}U(s)$$

(b) A PD or lead filter is needed. We can observe form the Bode plot 2 that the phase is dropping below -180 deg. To achieve good performances, one needs to raise the phase at the cross-over frequency, which requires a D term (or equivalently a lead filter). To avoid static error in the absence of disturbance, one needs the magnitude of the open-loop transfer function $L(s)$ to be unbounded at low frequencies, i.e. $\lim_{s\to 0} L(s) = \infty$. Since $G(s)$ has a pole at the origin, $\lim_{s\to 0} G(s) = \infty$, and it follows that the controller $F(s)$ does not require an integrator to remove the satic error. *Note: this would not be true in the presence of disturbances !*

(c) We first compute b using the fact that (see the cheat-sheet, bode plot of a lead filter):

$$\frac{\max_\omega |F(j\omega)|}{\min_\omega |F(j\omega)|} = b$$

for a lead filter. The best performance is achieved by using the largest admissible b, hence $b = 23\,\mathrm{db} \equiv 14.1$. The maximum phase of $F(j\omega)$ for this value of b is available from graph 3, and is $\varphi_{\max} = 60$ deg. It shall be placed at the cross-over frequency ω_c to get the best benefit. The cross-over frequency is then given by the formula (see the cheat-sheet):

$$\underbrace{\arg\{F(j\omega_{\mathrm{c}})\}}_{=\varphi_{\max}=60\,\mathrm{deg}} = -180 + \underbrace{\varphi_{\mathrm{margin}}}_{=60\,\mathrm{deg}} - \arg\{G(j\omega_{\mathrm{c}})\}.$$

An inspection of the Bode plot (2) shows that $\omega_{\mathrm{c}} = 10\,\mathrm{rad/s}$, and $|G(j\omega_{\mathrm{c}})| = 0\,\mathrm{db} \equiv 1$, hence:

1. using $\omega_{\mathrm{c}} = \frac{\sqrt{b}}{\tau_{\mathrm{d}}}$ it follows that $\tau_{\mathrm{d}} = \frac{\sqrt{14.1}}{10} = 0.38\,\mathrm{s}$
2. using $|F(j\omega_{\mathrm{c}})| = |G(j\omega_{\mathrm{c}})|^{-1}$ (see the cheat-sheet), it follows that $|F(j\omega_{\mathrm{c}})| = \sqrt{b}K_{\mathrm{p}} = 1$, and $K_{\mathrm{p}} = 0.27$.

2. **Poles, zeros, Bode & Nyquist**: consider a controller of the form $F(s) = K \cdot F_0(s)$, where $F_0(s)$ is stable. Controller $F(s)$ is applied to a system having a transfer function $G(s)$, which has a single unstable real pole. The closed-loop transfer function is stable for $K = 5$, with a modulus margin -7.5 db (see Figure 4, lower-right graph). Figure 4 charts the poles ($\times$) and zeros ($\circ$) of the closed-loop transfer function $T(s) = \frac{L(s)}{1+L(s)}$ (upper-left), of the open loop transfer function $L(s) = F(s)G(s)$ (for $K = 5$, upper-right), and the Bode plot and Nyquist plot of the open loop transfer function $L(s)$. Some values of the Bode plot are reported in Table 2.

   (a) (2 points) Compute the maximum of the module of the sensitivity function $S(s)$, i.e.:

   $$S_{\mathrm{m}} = \max_\omega |S(j\omega)|,$$

   justify your answer.

   (b) (2 points) Compute the range for the controller gain K that can be used without loosing the closed-loop stability of the system. Provide exact numbers. Justify.

   (c) (2 points) What is the order of the open-loop transfer function $L(s)$ (degree of the denominator) ? What is

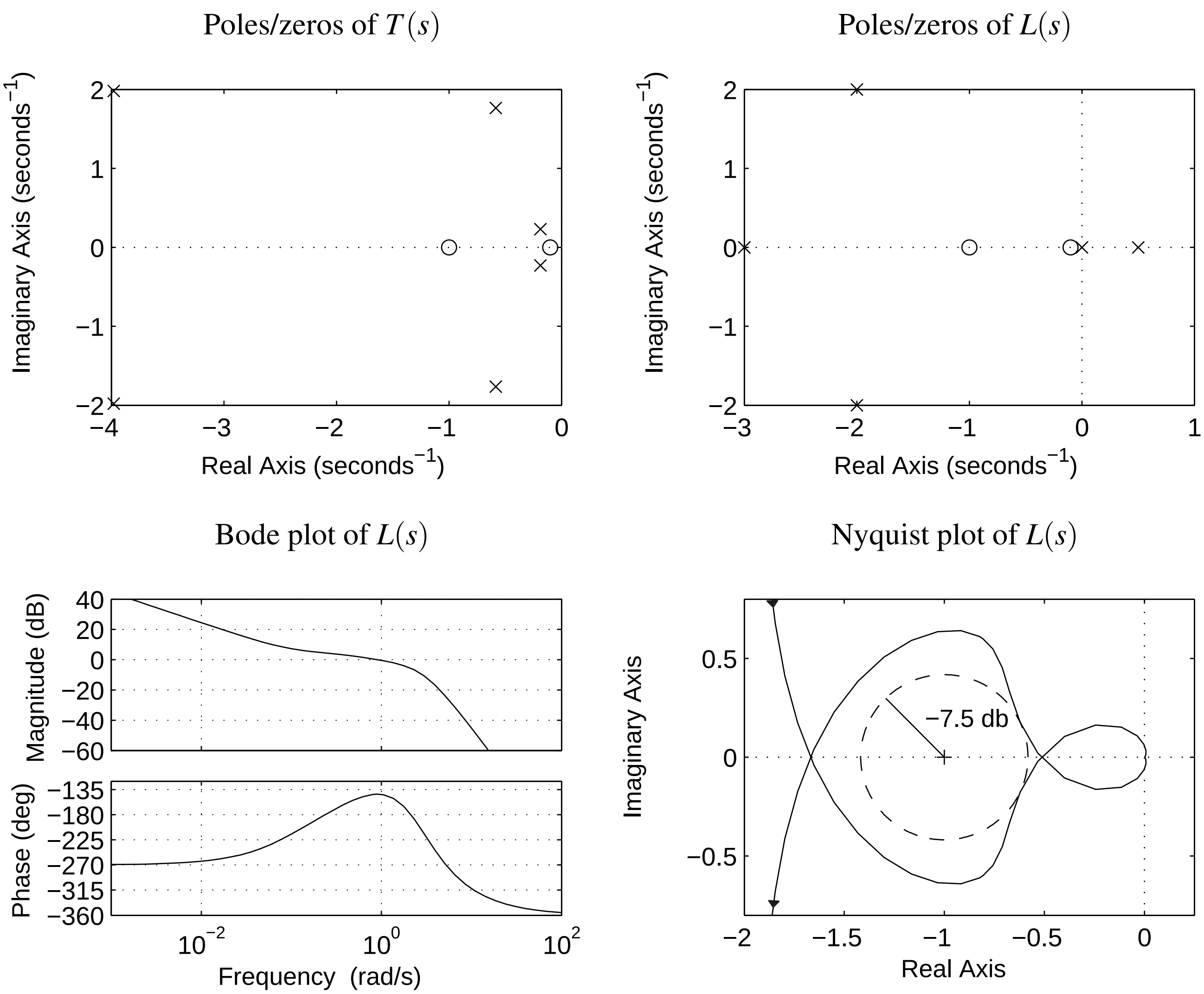


Figure 4: Poles and zeros of the closed-loop $T(s)$, of the open loop transfer function $L(s)$, Bode plot and Nyquist plot of the open loop transfer function $L(s)$.

Table 2: Specific values of the Bode plot of $L(s)$

Frequency (rad/s)	Magnitude (db)	Phase (deg)
0	∞	-270
0.24	4.45	-180
0.9	0.0	-143
2.12	-5.77	-180
∞	$-\infty$	-360

the degree of the numerator ? Is $G(s)$ minimum phase ? Justify.

Solution:

(a) We know that the maximum of the sensitivity function is given by:

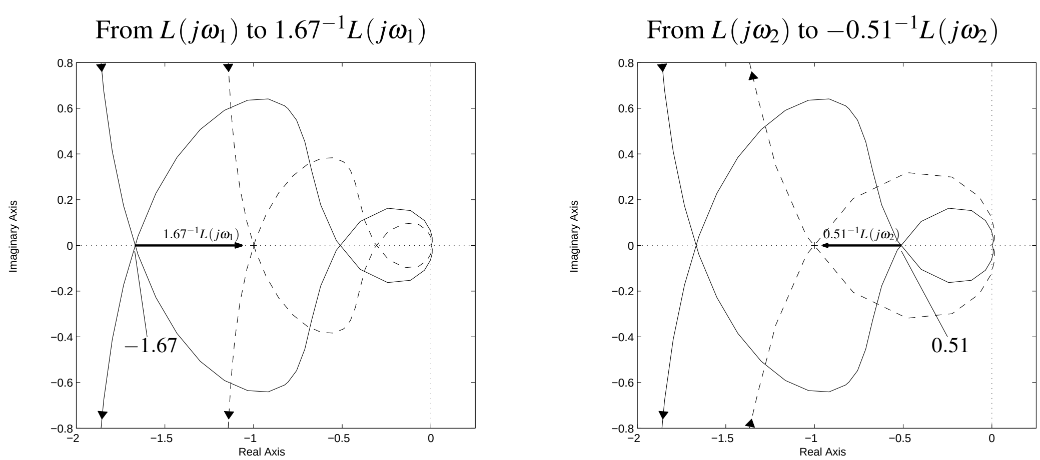
$$S_{\text{m}} = \max_{\omega} |S(j\omega)| = \max_{\omega} \frac{1}{|1+L(j\omega)|} = \left(\min_{\omega} |1+L(j\omega)|\right)^{-1}$$

which is the inverse of the minimal distance between the Nyquist curve and the critical point -1, or modulus margin. In our case, $S_{\text{m}} = 7.5\,\text{db} \equiv 2.37$.

(b) We observe from the Nyquist plot that the Nyquist curve crosses the real-axis (phase -180 deg) at the two frequencies $\omega_1 = 0.24$ and $\omega_2 = 2.12$, corresponding to the magnitudes $|L(j\omega_1)| = 4.45\,\text{db} \equiv 1.67$ and $|L(j\omega_2)| = -5.77\,\text{db} \equiv 0.51$ (the values are found in Table 2). It follows that:

$$1.67^{-1} L(j\omega_1) = 0.51^{-1} L(j\omega_2) = -1 \tag{4}$$

It is stated that the open-loop transfer function $L(s)$ has one unstable pole and the closed-loop system is stable for $K = 5$, hence the closed-loop system remains stable as long as the "main" loop encircles the critical point -1. The limits are provided by (4) (see Figure below for an illustration), i.e. the controller gain can range between $5 \cdot 1.67^{-1} = 3$ and $5 \cdot 0.51^{-1} = 9.8$.



(c) By inspection of the pole-zero map of the open-loop transfer function (upper-left graph in Fig. 4), we see that $L(s)$ has 5 poles and 2 zeros. Hence the degree of the numerator is 2 and the degree of the denominator is 5. We see that $L(s)$ does not have any zero in the right half plane (RHP), and we know that $F(s)$ is stable and therefore has no pole in the RHP. It follows that there can be no pole-zero cancellation in the RHP when $L(s) = F(s)G(s)$ is formed, hence the system $G(s)$ cannot have a zero in the RHP, and has to be minimum-phase.

3. **State-space**: consider the state-space representation:

$$\begin{aligned} \dot{X} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} U \\ Y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} X \end{aligned}$$

(a) (2 points) Is the system controllable ? Justify.

(b) (2 points) Is the system observable ? Justify.

(c) (2 points) Consider the state-space representation:

$$\begin{aligned} \dot{X} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U \\ Y &= \begin{bmatrix} 1 & 0 \end{bmatrix} X \end{aligned} \tag{5}$$

Compute the transfer function $G(s)$ that gives the input-output relationship $Y(s) = G(s) \cdot U(s)$. Is (5) a minimal realisation of $G(s)$? Justify.

Solution:

(a) The controllability matrix is formed by (see the cheat-sheet)

$$\mathscr{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & -1 \\ -2 & 0 & 0 \end{bmatrix}$$

The controllability can be assessed by computing the determinant of $\mathscr{C}$. However, it is faster to observe that the 2^{nd} and 3^{rd} columns of $\mathscr{C}$ are identical, hence $\mathscr{C}$ is not full column rank, and the system is not controllable.

(b) The observability matrix is formed by

$$\mathscr{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The observability can be assessed by computing the determinant of $\mathscr{O}$. However, it is faster to observe that the 3^{rd} row of $\mathscr{O}$ is zero, hence $\mathscr{O}$ is not full row rank, and the system is not observable.

(c) We apply the formula

$$G(s) = C(sI - A)^{-1} B = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{s-1}$$

We observe that we have made simplifications when forming $G(s)$, i.e.

$$\frac{s-1}{(s-1)^2} = \frac{1}{s-1}$$

hence (5) is not a minimal realisation. The same conclusion can be reached by noting that (5) is not observable.

4. **Theory**: (thoroughly justified answers are expected here)

(a) (2 points) Whenever the open-loop transfer function $L(s)$ has a relative degree higher or equal to 2, the Bode first integral states that:

$$\int_0^\infty \log|S(j\omega)|d\omega = \sum_{k=1}^{N} \mathrm{Re}\{p_k\} \tag{6}$$

where $p_1, ..., p_N$ is the collection of unstable poles of the open-loop transfer function $L(s)$, and $S = \frac{1}{1+L}$ is the sensitivity function of the closed-loop system. Explain the consequences of (6) for control. Moreover, in the light of (6), explain why and in what sense an unstable system is generally harder to control than a stable one. Use illustrative sketches to support your explanations if that helps.

(b) (2 points) For which type of system is it more difficult to achieve good control performances, minimum or non-minimum phase systems ? Why ?

(c) (2 points) Same question concerning delayed systems vs. non-delayed ones.

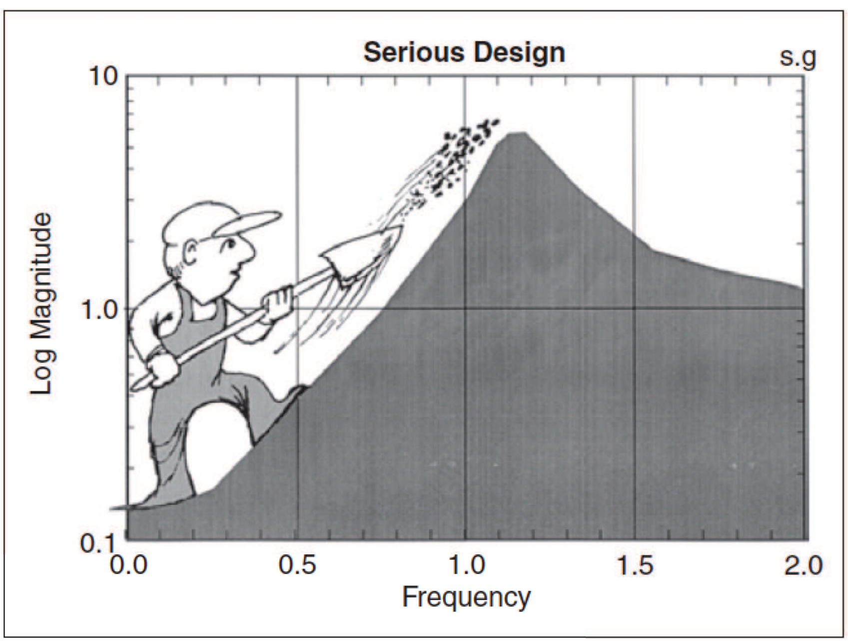


Figure 5: From ”Respect the Unstable”, G. Stein, IEEE Control System Magazine, 2003.

Solution:

(a) The Bode first integral is a statement on the balance between the area below 0db and above 0db in the log-linear plot of the magnitude of the sensitivity function $|S(j\omega)|$. In order to increase the closed-loop performance one needs to increase (or ”dig”) the area below 0db, and as a consequence increase the area above (Fig. 5 is an excellent illustration of that concept).

Generally, in control, we dislike large sensitivities. When the sensitivity is large, a) model errors have a strong impact on the closed-loop transfer function, b) the modulus margin, and therefore the robustness is poor.

For a stable system, the area above 0db matches the area below 0db. For an unstable system, however, the area above 0db in excess of the area below, to compensate for the unstable poles. This often increases dramatically the sensitivity above 0db and makes the controller harder to design.

(b) Non-minimum phase systems are harder to control, in the sense that the compromise performance-robustness is more critical than for minimum-phase systems. Non-minimum phase systems have zeros in the right-half plane (RHP), which result in an early and strong drop in the phase of the transfer function $G(s)$ of the system. The phase of $G(s)$ then crosses -180 deg early, and steeply, which can often not be pushed up effectively. Getting a reasonable phase margin then requires having a rather low cross-over frequency, hence a low bandwidth, and therefore relatively low control performances.

(c) If a delay is introduced in the transfer function $G(s)$, the resulting phase change between the un-delayed and and the delayed transfer function is $\arg\left\{e^{-s\delta}\right\}_{s=j\omega} = -\omega\delta$, where δ is the time delay in seconds. The phase drop is often difficult to handle for the reasons given in (b). The problem is made worse for a longer time delay δ because it increases (linearly) the phase drop at any given frequency.

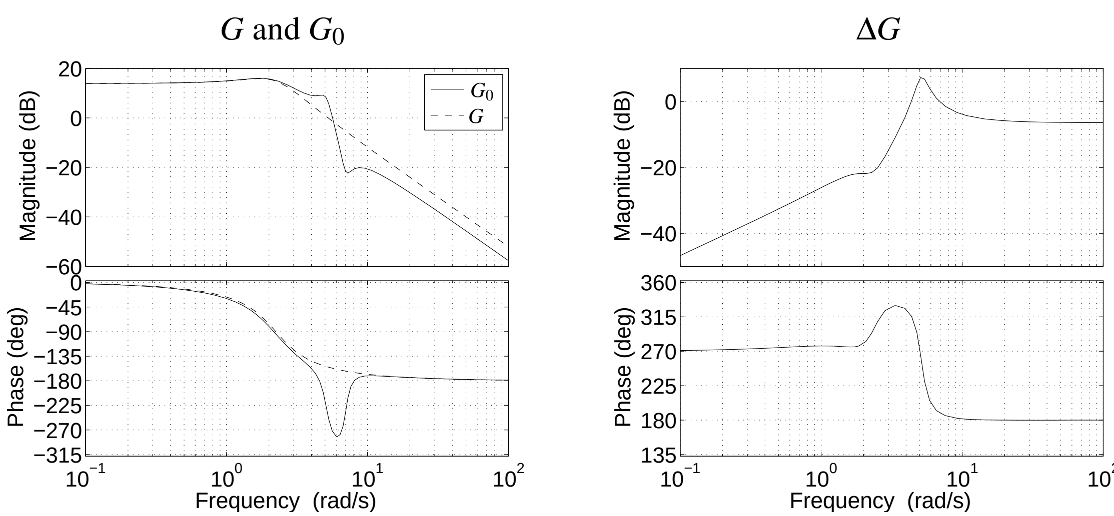


Figure 6: Bode plot of $G(s)$ and $G_0(s)$ (left graph). Bode plot of $\Delta G(s)$ (right graph).

5. **Miscellaneous topics**

(a) (2 points) Consider the transfer functions:

$$G_1(s) = \frac{s+1}{s+2}e^{-2s}, \qquad G_2(s) = \frac{s+1}{s-2}.$$

For both, compute the final value of the output $y(t)$, i.e. $\lim_{t\to\infty} y(t)$ if a unitary step is applied as an input to the system. Justify.

(b) (2 points) Consider the transfer function $G(s) = \frac{e^{-2s}}{(s+4)^2}$. An input signal

$$u(t) = 25\sin(3t) + e^{-0.5t}\cos(15t) + e^{-2.5t}\sin\left(5t + \frac{pi}{2}\right)$$

is applied to $G(s)$, compute the output $y(t)$ of the system after all transients have died out.

(c) (2 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{7}$$

compute the matrix exponential of A, i.e. e^A.

(d) (2 points) Consider a system whose dynamics are given by the transfer function $G_0(s)$ and modelled by the transfer function $G(s)$. The bode plots of $G(s)$, $G_0(s)$ and the model relative uncertainty $\Delta G(s)$ are given in Figure 6. Provide an upper-bound to the bandwidth (in rad/s) that can be achieved for this system. Justify and explain your assumptions.

(e) (2 points) Consider the open-loop transfer function:

$$L(s) = 2\frac{s+1}{s-1}. \tag{8}$$

Is the corresponding closed-loop transfer function stable ? Is it minimum-phase ? Justify.

(f) (2 points) Consider a mass suspended in levitation in the magnetic field developed by a electro-magnet (see

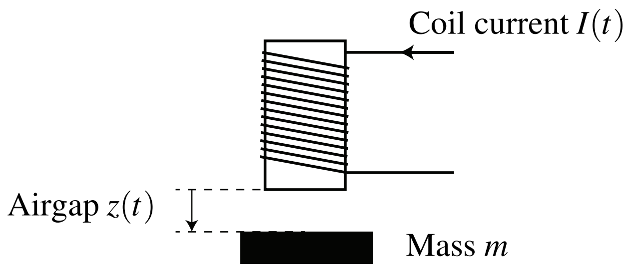


Figure 7: Sketch of the magnetic levitation.

Figure 7). The evolution of the airgap[2] $z(t)$ is given by:

$$m\ddot{z} = \frac{1}{2}\xi I^2 \frac{1}{(1+z)^2} - \mu\dot{z} - mg \tag{9}$$

where m is the mass, g is the gravity, ξ is a constant related to the physics of the system, μ a linear viscous friction coefficient, and I is the magnet current. Compute the transfer function from I to z, describing the system dynamics at the reference position $z = z_0$. We assume that $z_0 > 0$ and use only positive currents.

(g) (2 points) Give a <u>nonlinear</u> state-space representation for the system proposed in question (f)

Solution:

(a) The transfer function $G_1(s)$ is stable, hence the final value is given by

$$\lim_{t\to\infty} y_1(t) = \lim_{s\to 0} sG_1(s)\frac{1}{s} \tag{10}$$

where $\frac{1}{s}$ is the Laplace transform of a step function). $\lim_{t\to\infty} y_1(t) = G_1(0) = 0.5$. Transfer function $G_2(s)$ is unstable, hence its final value is unbounded.

(b) For $t \to \infty$ (after all transients have died), the input $u(t)$ tends to the signal $25\sin(3t)$. The output resulting from the purely sinusoidal input of frequency 3 then reads:

$$y(t) = 25\,|G(j3)|\sin(3t + \arg\{G(j3)\}) \tag{11}$$

where $25\,|G(j3)| = 1$ and $\arg\{G(j3)\} = -1.29\,\text{rad}$.

(c) Matrix A is nilpotent of degree 3 (i.e. it becomes zero for a power larger than 2), the easiest way to compute its exponential is then to form the series of the exponential:

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k \tag{12}$$

Since $A^3 = 0$, it follows that

$$e^A = I + A + \frac{1}{2}A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

(d) We will assume here that the closed-loop transfer function $T(s)$ decreases with a slope of $-20\,\text{db}/\text{dec}$. This assumption is in most cases fulfilled, indeed to get a good phase margin it is often required that the

[2] distance of the mass from the magnet

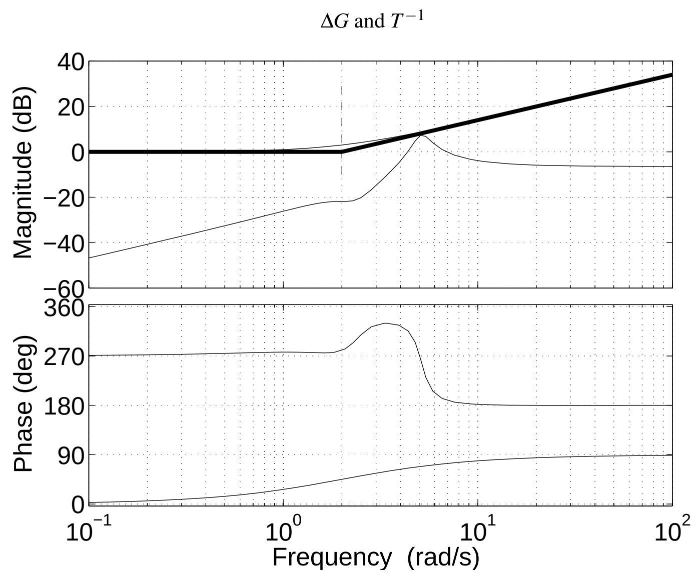
open-loop transfer function $L(s)$ passes the cross-over frequency at $-20\,\mathrm{db/dec}$ (phase-margin relationship), which yields the proposed assumption. Then we use here the robustness relationship:

$$|T(j\omega)| < |\Delta G(j\omega)|^{-1} \quad \forall\omega \tag{14}$$

ensuring the closed-loop stability despite model errors. For convenience, we use its inverse form:

$$|T(j\omega)|^{-1} > |\Delta G(j\omega)| \quad \forall\omega \tag{15}$$

A simple inspection in the graph (see below) shows that T can take its cut-off frequency at $\omega = 2\,\mathrm{rad/s}$ or less. This is the upper-bound for the closed-loop bandwidth.



(e) The closed-loop transfer function reads:

$$T(s) = \frac{L}{1+L(s)} = \frac{2(s+1)}{(s-1)+2(s+1)} = \frac{2(s+1)}{3s+1}$$

which has a pole at $-1/3$ and a zero at -1. The closed-loop transfer function is therefore minimum phase and stable.

(f) We first compute a steady-state point (I_0, z_0) for (9), using $\dot{z}_0 = \ddot{z}_0 = 0$:

$$\underbrace{m\ddot{z}_0}_{=0} = \frac{1}{2}\xi I_0^2 \frac{1}{(1+z_0)^2} - \underbrace{\mu\dot{z}_0}_{=0} - mg$$

which gives:

$$I_0 = \sqrt{2\xi^{-1}mg}\,(1+z_0)$$

We then linearize (9) at (I_0, z_0), i.e. :

$$\frac{\partial}{\partial\ddot{z}}[m\ddot{z}]_{z_0,I_0}\Delta\ddot{z} = \frac{\partial}{\partial z}\left[\frac{1}{2}\xi I^2 \frac{1}{(1+z)^2}\right]_{z_0,I_0}\Delta z + \frac{\partial}{\partial I}\left[\frac{1}{2}\xi I^2 \frac{1}{(1+z)^2}\right]_{z_0,I_0}\Delta I + \frac{\partial}{\partial\dot{z}}[-\mu\dot{z}]_{z_0,I_0}\Delta\dot{z}$$

giving

$$m\Delta\ddot{z} = -\left[\xi I^2 \frac{1}{(1+z)^3}\right]_{z_0,I_0} \Delta z + \left[\xi I \frac{1}{(1+z)^2}\right]_{z_0,I_0} \Delta I - \mu \Delta\dot{z}$$

Or equivalently, by replacing $z = z_0$ and $I = I_0$:

$$m\Delta\ddot{z} = -\left(\frac{2mg}{1+z_0}\right)\Delta z + \left(\frac{\sqrt{2\xi mg}}{1+z_0}\right)\Delta I - \mu\Delta\dot{z}$$

In the Laplace domain, this yields:

$$ms^2\Delta Z(s) = -\frac{2mg}{1+z_0}\Delta Z(s) + \frac{\sqrt{2\xi mg}}{1+z_0}\Delta I(s) - \mu s \Delta Z(s)$$

hence the transfer function:

$$\Delta Z(s) = \frac{\sqrt{2\xi mg}}{m(1+z_0)s^2 + \mu(1+z_0)s + 2mg}\Delta I(s)$$

(g) Using $v = \dot{z}$, equation (9) can be put in e.g. the nonlinear state-space form:

$$\begin{bmatrix} \dot{z} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ \frac{1}{2}\xi I^2 \frac{1}{(1+z)^2} - \mu v - mg \end{bmatrix}$$