Exam in the course Plasma physics with applications, RRY085

2012-10-23

Total number of points: 31
Grading: 50% correct answers and solutions for mark 3, 65 % for mark 4, and 80 % for mark 5.

Problem 1: Plasma descriptions (total points: 5)

Give short answers and explanations to the following questions and statements.

1. A plasma consists of a huge number of particles. Tracking the trajectory of each particle and keeping track of the interactions between all particles becomes impossible for systems with more than a couple of thousand particles. Therefore it is necessary to use a model of the plasma which takes account of *average* properties of the plasma. There are three main degrees of sophistication for these approximative descripitions. What are the three descriptions, or models, called? (1p)
2. Describe the main differences between these descriptions! (1p)
3. For investigating the properties of a wave in a cold unmagnetized plasma, which description should you use? (1p)
4. For investigating the stability properties of a Tokamak plasma, which description is most convenient? (1p)
5. When taking velocity moments of the Boltzmann equation, a set of equations is produced. Describe one serious problem with these equations, and suggest a way of overcoming this problem! (1p)

Problem 2: Fully ionized vs. partially ionized gases, collision and diffusion (total points: 9)

Give short answers and explanations to the following questions and statements.

1. Fusion plasmas and space plasmas are completely ionized, whereas most industrial applications of plasma uses partially, or weakly, ionized gases; where only a small fraction of the molecules and atoms are ionized. However, this minute degree of ionization has drastic implications for the electrical and chemical properties of the gas, which can be of great use in industrial applications. Name two applications of partially ionized gases! (1p)
2. Name two natural occurances of partially ionized gases, or plasma, on the earth, or at least inside the earth's atmosphere! (1p)
3. The collision dynamics of fully ionized and partially ionized gases are quite different. What are the different types of collisions, and what is the main difference? (1p)

4. Describe the ionization mechanism in a fully ionized plasma in thermal equilibrium! (1p)
5. Why is this ionization mechanism unusable for gases at atmospheric pressure and density, and what is the standard way of ionizing such gases? (1p)
6. Collisions lead to diffusion of particles. What is diffusion, and what does it lead to? (1p)
7. In the case of gases of very low ionization degree, the electrons diffuse freely, but for higher degrees of ionization, the interaction between ions and electrons becomes important. What is this type of diffusion called, and what is the main consequence for the diffusion rate? (1p)
8. In an unmagnetized plasma, the electron-ion collision frequency is determined by the electron mean free path. In a strongly magnetized plasma on the other hand, it is not the mean free path which is important, but rather a different length. Which length is this? (1p)
9. Give approximate formulas for the diffusion coefficient, D, in unmagnetized and strongly magnetized plasmas! (1p)

Problem 3: Current and heating (total points: 3)

1. In the presence of collisions, with ions or neutrals, the *average* motion of an electron in an unmagnetized plasma subject to a DC electric field is described by

   $$m\dot{\bar{v}} = -e\bar{E} - m\nu_c\bar{v}$$

   where m is the electron mass, $\bar{v}$ the electron velocity, dot denotes differentiation with respect to time, e is the electron charge, $\bar{E}$ the electric field, ν_c the electron collision frequency.

   After a short time, the electron velocity reaches a stationary value. This leads to a current. Given the definition of conductivity, $\bar{j} = \sigma\bar{E}$, where $\bar{j}$ is the current, and σ is the conductivity. What is the current and conductivity if the electron density is n (m^{-3})? (1p)
2. The current leads to absorption of energy from the electric field which is released in the plasma. Give an expression for the energy release per second per cubic meter in the plasma due to the current! (1p)
3. The inverse of conductivity is resistivity, η. In a fully ionized plasma, the resistivity decreases with temperature. Using the formula you found in the previous problem, explain why this a big problem for fusion applications! (1p)

Problem 4: MHD Equilibrium (total points: 6)

1. Describe **briefly** the two different cylindrical configurations (θ- and Z-pinch) that produce a pinching effect, i.e. a plasma density that peaks at the cylindrical axis. Discuss how the fields are generated and pinpoint the directions of the resulting fields and currents inside the plasma. Remember that a picture says more than a thousand words. (2p)

2. Derive the pressure balance expression for the θ-pinch: Use your conclusions from the previous problem to analyze the equilibrium MHD equations

$$\mu_0 \mathbf{J}_0 = \nabla \times \mathbf{B}_0 \tag{1}$$

and

$$\mathbf{J}_0 \times \mathbf{B}_0 = \nabla P_0 \tag{2}$$

by assuming cylindrical symmetry, $\partial/\partial\theta = 0$, and infinite cylinders, $\partial/\partial z = 0$! The curl in cylindrical coordinates reads (2p)

$$\nabla \times \mathbf{A} = \left(\frac{1}{R}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\widehat{R} + \left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right)\hat{\theta} + \frac{1}{R}\left(\frac{\partial (RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial \theta}\right)\hat{z} \tag{3}$$

3. Assume that the magnetic field in the z-pinch increases monotonically in radius from the the cylindrical axis and outwards to the plasma edge. Sketch the corresonding radial profiles for P_0 and $\mathbf{J}_0$. It is necessary to impose physical boundary conditions on $\mathbf{B}_0$. (2p)

Problem 5: Basic plasma properties (total points: 2)

The average particle separation in a fully ionized plasma is roughly $n^{-1/3}$, where n is the number density of electrons and ions. What is the potential energy between two electrons? Find a formula for the kinetic energy divided by the potential energy! A criterion for the ionized gas to truly be a plasma is that the number of particles in the Debye sphere should be very large, i.e. $\lambda_D^3 n \gg 1$. Using

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{n e^2}}$$

show that the kinetic energy is much larger than the potential energy! What are the implications of this for the plasma? (2p)

Problem 6: Single-Particle Motion (total points: 2)

The magnetic field due to an infinitely long wire carrying a current I is given by

$$\mathbf{B} = \frac{\mu_0 I}{2\pi R}\mathbf{e}_\theta \ , \tag{4}$$

where θ is the angle around the wire. Using single-particle theory, explain why this configuration is not able to confine individual charged particles (ions or electrons) in the (R, z)-plane! (2p)

Problem 7: MHD Normal Modes (Alfvén Waves) (total points: 4)

During the lectures, we found Alfvén waves in the low-frequency limit of waves that propagate parallel to the background magnetic field in a magnetized, cold plasma. Actually, we found two different types of Alfvén waves: The *compressional* Alfvén wave, corresponding to the low-frequency limit of the R-mode, and the *shear* Alfvén wave, corresponding to the low-frequency limit of the L-mode. Our derivation utilized the two-fluid equations. However, due to their low frequencies, Alfvén waves can be described much more easily using MHD theory (which indeed is suitable for low-frequency, large-scale instabilities). In this problem, we will rederive the dispersion relation for the shear Alfvén wave by means of linearized MHD.

The linearized MHD equation for a generic fluid velocity perturbation $\mathbf{V}_1 = \mathbf{R}(\mathbf{r})\,\mathrm{e}^{-i\omega t}$ in a plasma with equilibrum pressure P_0, equilibrium mass density ρ_{m0} and equilibrium magnetic field $\mathbf{B}_0$, is given by

$$-\omega^2 \rho_{m0}(\mathbf{r})\,\mathbf{R}(\mathbf{r}) = \mathbf{F}(\mathbf{R}(\mathbf{r})) \tag{5}$$

where the force operator is

$$\begin{aligned}\mathbf{F}(\mathbf{R}) = {} & \frac{1}{\mu_0}(\nabla\times\mathbf{B}_0)\times[\nabla\times(\mathbf{R}\times\mathbf{B}_0)] + \frac{1}{\mu_0}\{\nabla\times[\nabla\times(\mathbf{R}\times\mathbf{B}_0)]\}\times\mathbf{B}_0 \\ & + \nabla(\mathbf{R}\cdot\nabla P_0 + \gamma P_0 \nabla\cdot\mathbf{R})\end{aligned} \tag{6}$$

and ω is the frequency of the perturbation. For $\omega^2 < 0$, the perturbation grows exponentially in time, which corresponds to an instability, and for $\omega^2 > 0$, the pertubation oscillates, i.e. it is a wave. We are interested in the latter case.

1. Assume that the plasma equilibrium is infinite, homogeneous and incompressible (i.e. that the perturbed velocity satisfies $\nabla\cdot\mathbf{V} = 0$). Simplify the operator $\mathbf{F}$ accordingly! The RHS should now contain only one term. (1p)

2. Rotate the coordinate system so that $\mathbf{B}_0$ points in the z-direction and the wave vector is aligned according to $\mathbf{k} = k_\perp \mathbf{e}_y + k_\parallel \mathbf{e}_z$! Fourier analyze in space by setting

$$\mathbf{R}(\mathbf{r}) = \widetilde{\mathbf{R}}\,\mathrm{e}^{i\left(k_\perp y + k_\parallel z\right)} \tag{7}$$

For shear waves, only the component perpendicular to the propagation direction, i.e. the wave vector $\mathbf{k}$, is nonvanishing. Hence, $\widetilde{R}_y = \widetilde{R}_z = 0$. Under these assumptions, derive the dispersion relation for the shear Alfvén wave by solving (5)! (3p)