MPWPS, year 1, study period 1, academic year 2013/2014

Exam
Electromagnetic Waves and Components (RRY 036), 25/10 2013
Department of Earth and Space Sciences

Teachers:
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On the exam you may use (indexations and markings in formulas are permitted):
- Chalmers-approved calculator,
- Formulas in Electromagnetic waves (E. Palmberg 2012),
- Formulae and constants for blackbody radiation, excitation of two-level systems and radiative transfer (A. Heikkilä 2013),
- Physics Handbook, Beta.
- Dictionary (not electronic).

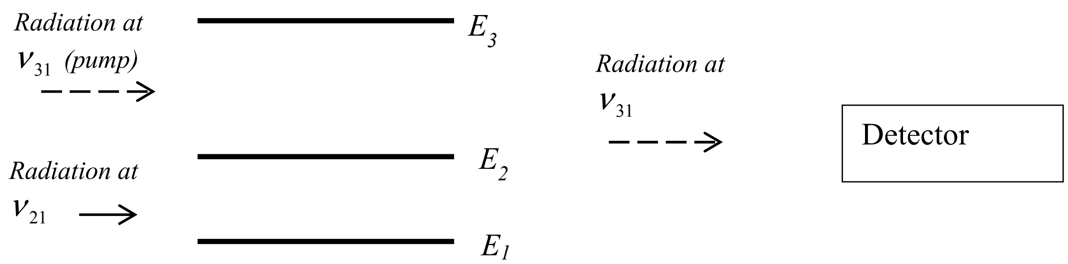
Grade limits:
Grade 3 (=pass): 20 points
Grade 4: 30 points
Grade 5: 40 points
A maximum of 50 points can be achieved on the exam.

Solutions to the exam will be available on the course homepage no later than 26/10 2013. The time and date for review of the examination will be published on the course homepage.

Remember: Give full solutions to the problems you hand in, i.e. explain and motivate your answers carefully! Be careful with units! When drawing graphs, indicate clearly the quantity on each axis, and give the scale.

1. A black body radiator is in the form of a cavity with a small opening. Consider radiation emitted straight forward through the opening. Derive a formula for the *peak value* of the specific intensity as function of the temperature. Sketch a graph. Set a scale by marking the specific intensity at 10^4 K. (2p)

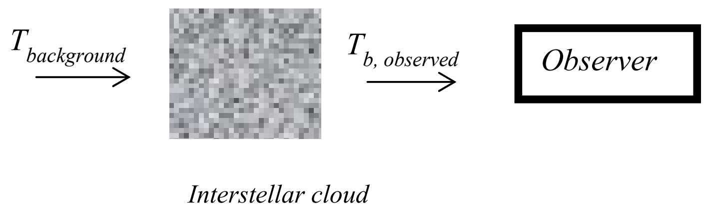
2. Study the 3-level system below. A "pump" exposes the system to radiation at the frequency $\nu_{31} = \frac{E_3 - E_1}{h}$ thereby exciting particles from level 1 to level 3. Fast non-radiative processes move the particles from level 3 to levels 2 and 1. Since the pump rate is high, and assuming a slow spontaneous emission between 2 and 1, most of the particles in this "excitation cycle" will end up at level 2, leaving level 1 almost empty. Hence the system becomes transparent for the pump-radiation. A detector measures the intensity of the pump-radiation at ν_{31} passing through the system.



If another source of radiation resonant with the levels 2 & 1 ($\nu_{21} = \frac{E_2 - E_1}{h}$) is switched on, *will the intensity at ν_{31} measured by the detector increase, decrease or stay constant?* Motivate your answer carefully! (2p)

3. The ^{13}CO molecule has rotational transitions in the mm-wavelength region. The ground state transition at 110,2 GHz has an Einstein *A*-coefficient of $6{,}3 \cdot 10^{-8}$ s^{-1}, statistical weights $g_l = 1$ and $g_u = 3$, and a collision coefficient (c_{ul}) of $3{,}3 \cdot 10^{-11}$ cm^3 s^{-1}.

An observer makes measurements of EM-radiation originating from the sky at frequencies around 110,2 GHz. Between the observer and a background source is an interstellar gas cloud, mainly composed of molecular hydrogen and atomic helium, mixed with very small amounts of other molecules, like ^{13}CO, ^{12}CO, C^{18}O, HCN, CS, HCO$^+$ etc.



Background source: 60 kelvin black body.
Interstellar cloud: number density $5{,}0 \cdot 10^4$ cm^{-3}, kinetic temperature 30 K, optical depth (at line centre) 0,3.
You may treat ^{13}CO as a two-level system and neglect the influence of absorption and stimulated emission. *Sketch a spectrum of the observed radiation (brightness temperature as function of frequency) around 110,2 GHz.* (3p)

4. Consider plane electromagnetic waves in a collisionless plasma.

a) Give the permittivity $\varepsilon(\omega)$ for a simple model of a collisionless plasma, starting from the expression for $\varepsilon(\omega)$ for the harmonic oscillator model. (1p)

b) Derive the dispersion relation giving $\omega=\omega(k)$ for electromagnetic waves in the plasma. (3p)

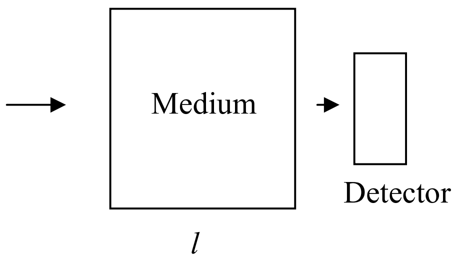
c) Calculate the phase velocity v_p and the group velocity v_g in the plasma. Explain the concepts phase velocity and group velocity in a sketch. (4p)

d) What happens to waves with $\omega<\omega_p$? Motivate your result by solving the dispersion relation. (2p)

5. The conductivity σ of a lossy medium can be measured through microwave absorption using the set-up in the figure. A 1 GHz TEM plane wave is sent normally through the medium and the power transmitted through the medium is measured. The medium has $\mu=\mu_0$ and $\varepsilon_d=12\varepsilon_0$.

a) List all the effects that can make the power received by the detector less than that incident on the medium. (2p)

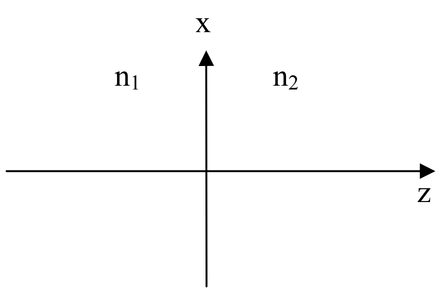
b) For a sample thickness l=0.5 mm, the power reaching the detector is 2 mW/cm^2. If the sample thickness is increased to l=1 mm, the power at the detector drops to 0.1 mW/cm^2. Use this information to compute the conductivity σ of the medium. Assume that the medium is a good conductor at this frequency. (6p)



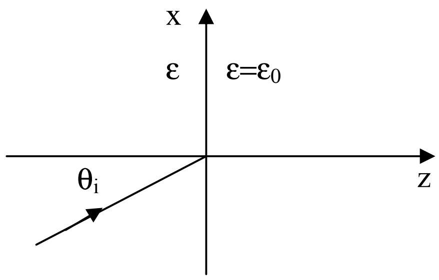
6. A 1 THz plane electromagnetic wave with $\underline{\mathbf{E}}_i=\hat{x}E_0e^{-j\beta z}$ is normally incident from a dielectric medium with refractive index $n=n_1$ into a dielectric medium with $n=n_2$ (both media are lossless with $\sigma=0$ and $\mu=\mu_0$).

a) Determine the ratio n_2/n_1 so the reflected and transmitted power are the same. (5p)

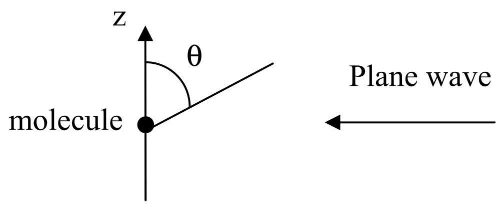
b) Assume that medium 1 is air and medium 2 fulfils the requirement in a) with $n_2/n_1>1$. Design an antireflective coating in order to reduce as much as possible of the reflection from medium n_2. The thickness has to exceed 100 μm for this particular application. What parameters would you choose for the coating (thickness, refractive index)? Motivate your design with calculations! (5p)



7. A plane wave with $\mathbf{E}(\mathbf{r},t)=8\cos(\omega t-2x-4z)\hat{y}$ V/m is obliquely incident on an interface between a dielectric with $\varepsilon=12\varepsilon_0$, $\mu=\mu_0$ and $\sigma=0$, and air.
a) Determine the polarization of the wave (TE or TM). (1p)
b) Determine the frequency of the wave. (3p)
c) Determine the reflected field $\mathbf{E}_r(\mathbf{r},t)$ and the k-vector in air ($z\geq0$). (6p)



8. A plane wave of unpolarized light with frequency f=500 THz and intensity 100 W/m^2 is incident upon a molecule with resonance angular frequency $\omega_0=2\cdot10^{16}$ rad/s.
a) Calculate the total re-radiated power by the molecule and discuss its distribution in angle θ in the plane indicated in the figure. The classical electron radius is $r_e=2.82\cdot10^{-15}$ m. (3p)
b) Discuss the polarization of the re-radiated wave for $\theta=0°$. (2p)



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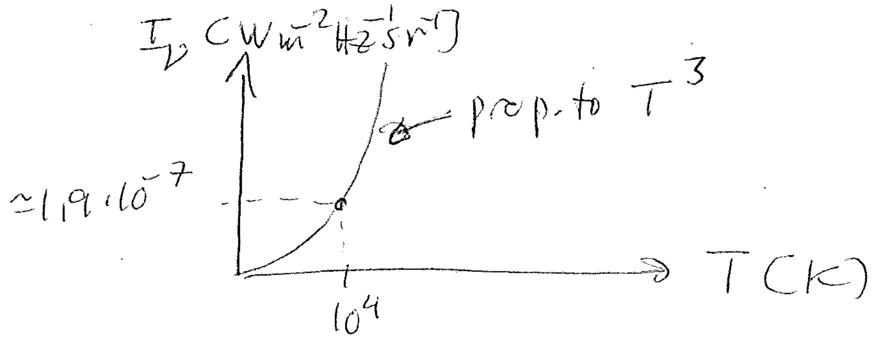
(1) $I_\nu = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT}-1}$

$\nu_{peak} \simeq 5{,}88 \cdot 10^{10} T \Rightarrow I_{\nu,peak} \simeq 1{,}896 \cdot 10^{-19} T^3$



(2) Radiation at ν_{21} : stim. em. takes particles from level 2 to level 1.

$\Rightarrow$ Radiation at ν_{31} is partially absorbed since level 1 now populated $\Rightarrow$ Less of ν_{31}-radiation passes $\Rightarrow$ decreased intensity at detector.

(3) $T_b \approx T_{bg}$ outside the spectral line

$T_b = T_{bg} e^{-\tau_\nu} + T_{ex} \cdot (1 - e^{-\tau_\nu})$ at line-centre.

stat. equil.:

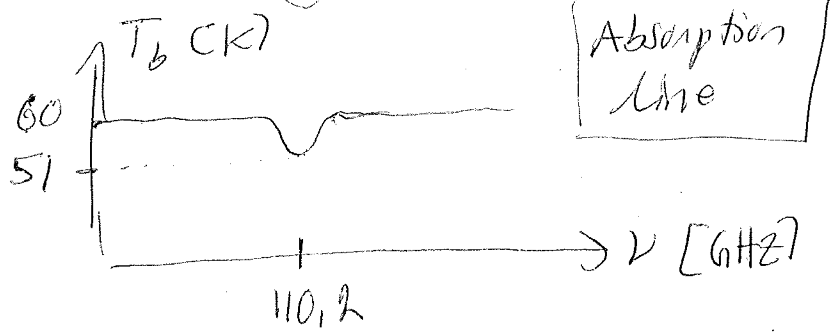
$$\frac{n C_{lu}}{n C_{ul} + A_{ul}} = \frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-h\nu_{ul}/kT_{ex}}$$

use $n_{crit} = \frac{A_{ul}}{C_{ul}}$; $\frac{C_{lu}}{C_{ul}} = \frac{g_u}{g_l} e^{-h\nu_{ul}/kT_{kin}}$

$$\Rightarrow \frac{e^{-h\nu_{ul}/kT_{kin}} \cdot n}{n + n_{crit}} = e^{-h\nu_{ul}/kT_{ex}}$$

$\Rightarrow T_{ex} \simeq 24{,}74$ K

$\Rightarrow T_{b_0} \simeq 51$ K $\Rightarrow$



4 a) $\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)$ ($\omega_0 = \gamma = 0$ in harm. oscillator model)

b) Dispersion relation:

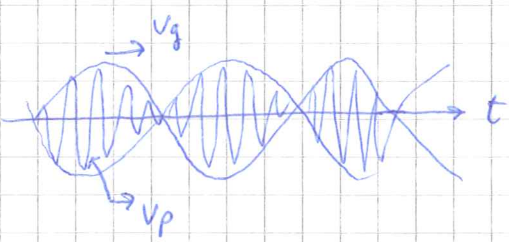
$$k = \frac{\omega}{c_0} \sqrt{\varepsilon/\varepsilon_0}$$

$$k^2 = \frac{1}{c_0^2}\left(\omega^2 - \omega_p^2\right) \Rightarrow \omega^2 = c_0^2 k^2 + \omega_p^2$$

c) $$v_p = \frac{\omega}{k} = \frac{\omega\, c_0}{\sqrt{\omega^2 - \omega_p^2}} = \frac{c_0}{\sqrt{1 - \omega_p^2/\omega^2}}$$

$$v_g = \frac{\partial \omega}{\partial k} = \frac{1}{\partial k/\partial \omega} = c_0 \sqrt{1 - \omega_p^2/\omega^2}$$

~~d~~)



d) $\omega < \omega_p \Rightarrow k = \frac{1}{c_0}\sqrt{\omega^2 - \omega_p^2} = -j\alpha$

Wave is not propagating ($\beta = 0$), damped as $e^{-\alpha z}$

5 $\mu = \mu_0$, $\sigma = ?$
$\varepsilon = 12\varepsilon_0$

a) Reflection from surface 1 and 2.
Absorption in medium between surface 1 and 2.
small absorption in air

b) Reflection is the same in the two cases ($\ell_1 = 0.5$ mm, $\ell_2 = 1$ mm).
The difference in received power is due to absorption:

$$\frac{P_2}{P_1} = e^{-2\alpha\Delta\ell} = \frac{0.1}{2}, \quad \Delta\ell = 0.5\cdot 10^{-3}\ \text{m}$$

$$\Rightarrow -2\alpha\Delta\ell = \ln 0.05$$

$$\Rightarrow \alpha = -\frac{\ln 0.05}{2\Delta\ell} = 2995.7\ \text{m}^{-1}$$

Good conductor: $\alpha = \sqrt{\frac{\mu_0\omega\sigma}{2}}$

$$\alpha^2 = \frac{\mu_0\omega\sigma}{2} \Rightarrow \sigma = \frac{2\alpha^2}{\mu_0\omega} = \frac{2\cdot(2995.7)^2}{4\pi\cdot 10^{-7}\cdot 2\pi\cdot 10^9} =$$

$$= \quad \Rightarrow \quad \sigma = 2.27\cdot 10^3\ \text{S/m}$$

Check: $\frac{\sigma}{\omega\varepsilon} = \frac{2.27\cdot 10^3}{2\pi\cdot 10^9\cdot 8.85\cdot 10^{-12}} \gg 1$, good conductor

Due to absorption, multiple reflections give very small contribution.

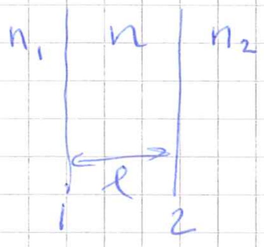
6. a) $R = \left|\frac{n_1-n_2}{n_1+n_2}\right|^2$, $T = 1-R$

$T = R \Rightarrow R = \frac{1}{2}$

$\Rightarrow \frac{n-n_2}{n_1+n_2} = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow \frac{n_2}{n_1} = 3 \pm 2\sqrt{2} = \begin{cases} 5.82 \\ 0.17 \end{cases}$

b) $n_1 = 1$, $n_2 = 5.82$



Choose $l = \frac{\lambda}{4}(2m+1)$ thickness with $n = \sqrt{n_1 n_2} = 2.4$

Proof: see orfanidis: $Z_1 = \frac{\eta^2}{Z_2}$, $Z_2 = \eta_2$, $\Gamma_1 = \frac{Z_1-\eta_1}{Z_1+\eta_1}$ <...

$\lambda = \frac{\lambda_0}{n}$, $\lambda_0 = \frac{3\cdot 10^8}{10^{12}} = 300\cdot 10^{-6}$ m

$\frac{\lambda_0}{4n} = 31.25\cdot 10^{-6}$ m

Choose $m = 2$, $l = 5\cdot\frac{\lambda}{4} = 156.25\cdot 10^{-6}$ m

7 $\vec{E}(\vec{r},t) = 8\cos(\omega t - 2x + 4z)\,\hat{y}$

$\Rightarrow k_x = 2\ m^{-1}$, $k_z = 4\ m^{-1}$, $k_y = 0$

$\varepsilon = 12\,\varepsilon_0$, $\mu = \mu_0$, $\sigma = 0$

a/ TE polarized wave

b) Dispersion relation:

$$k^2 = \frac{\omega^2}{c_0^2} n^2$$

$$k_x^2 + k_z^2 = \frac{\omega^2}{c_0^2} \cdot 12 \Rightarrow \omega = 3.87 \cdot 10^8\ rad/s$$

$$\Rightarrow f = \frac{3.87 \cdot 10^8}{2\pi} = 6.16 \cdot 10^7\ Hz$$

c) We have $\theta_c = a\sin\frac{1}{\sqrt{12}} = 16.8^\circ$

i.e. $\theta_i > \theta_c \Rightarrow$ total reflection

$$\rho_{TE} = \frac{\sqrt{12}\cos\theta_i - 1\cdot\cos\theta_t}{\sqrt{12}\cos\theta_i + 1\cdot\cos\theta_t}$$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = -j\sqrt{\sin^2\theta_t - 1}$$

$\sin\theta_t = \sqrt{12}\sin\theta_i$ (Snell's law)

$$\Rightarrow \rho_{TE} = \frac{3.08 + j\,1.18}{3.08 - j\,1.18} = e^{j\,41.9^\circ}$$

$$\Rightarrow \vec{E}_r(\vec{r},t) = \hat{y}\,8\cos(\omega t - 2x + 4z + 41.9^\circ)\ V/m$$

For $z \geq 0$ we have the dispersion rel. (air):

$$k_x'^2 + k_z'^2 = \frac{\omega^2}{c_0^2}$$

$k_x' = k_x = 2\ m^{-1}$ (Snell's law)

$$k_z'^2 = \frac{\omega^2}{c_0^2} - k_x^2 = 1.66 - 4 = -2.3\ m^{-1}$$

$$\Rightarrow k_z = -j\alpha = -j\,1.52\ m^{-1}$$

8. a) $P_{rad} = \sigma P_{in}$

$$\sigma = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{\omega_0^4}$$ Rayleigh scattering $\omega << \omega_0$

$$\sigma = 6.66\cdot 10^{-29} \frac{\omega^4}{\omega_0^4}\ m^2$$

$\omega = 2\pi\cdot 10^{12}\cdot 500$, $\omega_0 = 2\cdot 10^{16}$, $\left(\frac{\omega}{\omega_0}\right)^4 = 6.1\cdot 10^{-4}$

$\Rightarrow \sigma = 4.06\cdot 10^{-32}\ m^2$

$\Rightarrow P_{rad} = 4\cdot 10^{-30}\ W$

This is the sum of the TE + TM powers.

Angular distribution:

50% TM:

50% TE

b) For $\theta = 0°$, the TM polarisation will not give any radiation, i.e. only TE polarised radiation for $\theta = 0$ (and $\theta = 180°$).