TMA690
Matematik Chalmers
Tentamensskrivning i Partiella diferentialekvationer F3 / TM3
Datum: 2017-01-12, kl. 8:30 - 12:30
Jjälpmedel: Inga
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1. Lös ekvationen

$$yu_x + xu_y = 0,$$

då $u(x, 2x) = x^2$. (6p)

2. Låt Ω vara enhetsdisken i $\mathbb{R}^2$. Ge en variationsformulering till problemet

$$\begin{aligned} -\Delta u + u &= f(x) && \text{i} \quad \Omega \\ \frac{\partial u}{\partial n} &= g && \text{på} \quad \partial\Omega. \end{aligned}$$

Verifiera sedan villkoren i Lax-Milgrams sats för variationsproblemet.
. (7p)

3. Bevisa att den enda lösningen $u \in \mathcal{C}^2(D)$ till

$$\begin{aligned} -\Delta u + u^3 &= 0 && \text{i} \quad D := x^2 + y^2 < 1 \\ u &= 0 && \text{på} \quad \partial D := x^2 + y^2 = 1. \end{aligned}$$

är $u = 0$. (6p)

4. Låt D vara enhetsdisken i $\mathbb{R}^2$ och antag att u satisfierar

$$\begin{aligned} &u_t = \Delta u \quad \text{i} \quad D \times [0, T], \\ &u(t, x, y) = 0 \quad \text{på} \quad \partial D \times [0, T]. \end{aligned}$$

Visa att

$$\int_D |\nabla u(t, x, y)|^2 dxdy \leq \int_D |\nabla u(0, x, y)|^2 dxdy$$

för $t \in [0, T]$. Här är $\nabla = (\partial_x, \partial_y)$. (6p)

5. Lös ekvationen

$$\begin{aligned} &u_{tt} = u_{xx}, && (t \geq 0, -\infty \leq x \leq \infty), \\ &u(0, x) = \sin(x), && (-\infty \leq x \leq \infty), \\ &u_t(0, x) = x^2, && (-\infty \leq x \leq \infty), \end{aligned}$$

(6p)

6. Låt

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} \phi(y) dy,$$

där $\phi = \phi(x)$ är begränsad och kontinuerlig för $-\infty < x < \infty$. Alltså u är en lösning till värmeledningsekvationen. Visa att $\lim_{t\to 0} u(t,x) = \phi(x)$ för alla x. (7p)

7. Bevisa att om $u \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$ inte är konstant, och u är harmonisk på D, så antar u sitt maximum och minimum på ∂D. (6p)

8.
a) Beskriv skillnaden mellan en distribution och en tempererad distribution. (3p)

b) Formulera Weierstrass approximationssats och förklara med ord huvudidén i beviset. (3p)

Betygsgränser: 20-29 p ger betyget 3; 30-39 p ger betyget 4; 40-50 p ger betyget 5.

Lycka till!/Sebas

① $y u_x + x u_y = 0$, $u(x, 2x) = x^2$

$\frac{dy}{dx} = \frac{x}{y}$, then $y\,dy = x\,dx$

so that $y^2 - x^2 = c$, and we use

$$\begin{cases} \xi = x \\ \eta = y^2 - x^2 \end{cases} \Rightarrow \begin{matrix} \xi_x = 1 & \xi_y = 0 \\ \eta_x = -2x & \eta_y = 2y \end{matrix}$$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi - u_\eta 2x$

$u_y = u_\xi \xi_y + u_\eta \eta_y = 0 + u_\eta 2y$

$\Rightarrow y u_x + x u_y = y u_\xi = 0 \Rightarrow$

$\underline{u_\xi = 0} \Rightarrow u = f(\eta) \Rightarrow u(x,y) = f(y^2 - x^2)$

$u(x,2x) = f((2x)^2 - x^2) = f(3x^2) = x^2$

$\Rightarrow f(t) = \frac{t}{3} \Rightarrow u(x,y) = f(y^2-x^2) = \frac{y^2-x^2}{3}$

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② $$\left.\begin{matrix} -\Delta u + u = f & \text{i } \Omega \\ \frac{\partial u}{\partial n} = g & \text{på } \partial\Omega \end{matrix}\right\}$$

We multiply by $v \in H^1(\Omega)$, ~~and~~ integrate over Ω

$$-\int_\Omega \Delta u v\,dx + \int_\Omega u v\,dx = \int_\Omega f v\,dx$$

Now, integrating by parts (Green's formula) we get

$$-\int_\Omega \Delta u v\,dx = \int_\Omega \nabla u \nabla v\,dx - \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial n}}_{=g} \cdot v\,ds$$

so we have: Find $u \in H^1(\Omega)$ such that

$$(V)\quad \int_\Omega \nabla u \nabla v\,dx + \int_\Omega u v\,dx = \int_\Omega f v\,dx + \int_{\partial\Omega} g v\,ds, \quad \forall v \in H^1(\Omega)$$

Now we verify the conditions for the Lax-Milgram theorem, being

$$a(v,w) = \int_\Omega (\nabla v \nabla w + vw)\, dx$$

$$L(w) = \int_\Omega f w \, dx + \int_{\partial\Omega} g w \, ds$$

1) a is symmetric.
2) $a(v,v) = \|v\|^2_{H^1(\Omega)}$, so V-elliptic
3) $|a(v,w)| \leq \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}$, so a is continuous using Cauchy-Schwarz in $H^1(\Omega)$.
4) Using Cauchy-Schwarz in $L_2(\Omega)$

$$\left|\int_\Omega f v \, dx\right| = |(f,v)_{L_2(\Omega)}| \leq \|f\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)}$$
$$\leq \|f\|_{L_2(\Omega)} \|v\|_{H^1(\Omega)}$$

Using Cauchy-Schwarz in $L_2(\partial\Omega)$ and trace inequality

$$\left|\int_{\partial\Omega} g v \, ds\right| \leq \|g\|_{L_2(\partial\Omega)} \|v\|_{L_2(\partial\Omega)}$$

$$\leq \|g\|_{L_2(\partial\Omega)} \cdot C \|v\|_{H^1(\Omega)}$$

Putting them together we get

$$|L(v)| \leq \left(\|f\|_{L_2(\Omega)} + C\|g\|_{L_2(\partial\Omega)}\right) \cdot \|v\|_{H^1(\Omega)}$$

③ $\begin{cases} \Delta u = u^3 & \text{i } D \\ u = 0 & \text{på } \partial D \end{cases}$

$$-\int_D \Delta u\, v\, dx + \int_D u^3 v\, dx = 0 \quad \Rightarrow$$

$$\int_D \nabla u \nabla v\, dx - \int_{\partial D} \frac{\partial u}{\partial \vec{n}} v \cdot ds + \int_D u^3 v\, dx = 0$$

Using $v = u$, and $v = u = 0$ on ∂D, we have

$$\int_D |\nabla u|^2 dx - 0 + \int_D u^4 dx = 0$$

but both terms are ≥ 0, so it is true only if they are identically zero, i.e, $\int_D u^4 = 0 \Rightarrow u = 0$

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④ $\begin{cases} u_t = \Delta u & \text{i } D \times [0,T] \\ u(t,x,y) = 0 & \text{på } \partial D \times [0,T] \end{cases}$

Define $E(t) = \int_D |\nabla u(t,x,y)|^2 dxdy$

We can see that $\frac{d}{\partial t} E(t) \leq 0$ as follows

$$\frac{d}{\partial t} E(t) = 2\int_D \nabla u \cdot \nabla u_t\, dxdy = \left\{ \begin{matrix} \text{integration} \\ \text{by parts} \end{matrix} \right\}$$

$$-2\int_D \Delta u\, u_t\, dxdy + \underbrace{2\int_{\partial D} \frac{\partial u}{\partial \vec{n}} \cdot u_t\, ds}_{=0\ (u_t = 0 \text{ on } \partial D \times [0,T])} =$$

$$= -2\int_D \Delta u\, u_t\, dxdy = -2\int (\Delta u)^2 dxdy \leq 0$$

so $E(t)$ is decreasing, so that $E(t) \leq E(0)$

(5)

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,x) = \sin(x) \\ u_t(0,x) = x^2 \end{cases}$$

Using d'Alembert's solution

$$u(x,t) = \frac{1}{2}\left[\sin(x-t) + \sin(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} y^2\,dy$$

$$= \frac{1}{2}\sin(x-t) + \sin(x+t) + x^2 t + \frac{1}{3}t^3$$

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(6) Second part of theorem about existence of the solution for the heat equation.

(7) (Strong Maximum principle for Harmonic functions), (book. theorem 28)

(8) a) theory (Peter Sjögren notes)

b) $f(x) \in \mathcal{C}([a,b])$, then for every $\varepsilon > 0$ $\exists\, p(x)$ polynomial such that

$$\max_{a \le x \le b} |f(x) - p(x)| < \varepsilon .$$

(proof in the book, theorem 19.)