TMA690 Partiella Differentialekvationer

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Lecture notes, and solutions to a selection of homework problems.

1 Lecture 2017.10.30

Notation: A multi index α is a vector in $\mathbb{R}^d$ whose components α_j are non-negative integers. The length $|\alpha|$ of α is defined by

$$|\alpha| = \sum_{j=1}^{d} \alpha_j.$$

If $v : \mathbb{R}^d \to \mathbb{R}$ we may use the multi index notation to define partial derivatives of order $|\alpha|$:

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_k^{\alpha_k}}.$$

Example: $\alpha = (1, 0, 1),\ |\alpha| = 2$

$$D^\alpha v = \frac{\partial^2 v}{\partial x_1 \partial x_3}.$$

Notation: For $\xi \in \mathbb{R}^d$ we define $\xi^\alpha = \xi_1^{\alpha_1} \cdot ... \cdot \xi_d^{\alpha_d}$.

Example: $\alpha = (1, 0, 1),\ \xi = (\xi_1, \xi_2, \xi_3) \Rightarrow \xi^\alpha = \xi_1 \cdot \xi_3$.

In this course we will mainly consider linear partial differential equations of the form

$$\alpha u = \alpha(x, D)u = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u = f, \text{ in } \Omega$$

Ω is an open connected set.

Definition: We say that the direction $\xi \in \mathbb{R}^d$, $\xi \neq 0$, is a characteristic direction for the operator $\alpha(x, D)$ at x if

$$\Lambda(\xi) = \Lambda(\xi, x) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha = 0.$$

Note: in the sum we only take $|\alpha| = m$ (principle part).

Definition: A $(d-1)$-dimensional surface is given locally as a function $F : \mathbb{R}^d \to \mathbb{R} \quad F(x_1, ..., x_d) = 0$. The normal is given as $\nabla F = (\frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n})$ for $x \in \mathbb{R}^d$ on surface.

Main Examples:

Example: First order scalar equations:

$$\sum_{j=1}^{d} a_j(x) \frac{\partial u}{\partial x_j} + a_0(x) u = f, \qquad \left(\sum_{|\alpha| \le 0} a_\alpha(x) D^\alpha u = f \right)$$

Characteristic equation:

$$\sum_{j=1}^{d} a_j(x) \cdot \xi_j = 0 \qquad \left(\sum_{|\alpha| = 1} a_\alpha(x) \xi^\alpha = 0 \right)$$

Then ξ is a characteristic direction if ξ is perpendicular to $(a_1(x), ..., a_d(x))$.

Example: Let

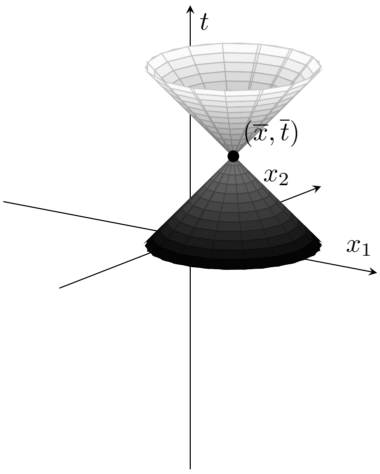
$$\Delta u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}$$

Poisson's equation: $-\Delta u = f$. Characteristic equation $\Lambda(\xi) = -(\xi_1^2 + ... + \xi_d^2) = 0 \Rightarrow \xi = 0$. This means that there are no characteristic directions.

Example: Heat equation $\frac{\partial u}{\partial t} - \Delta u = f$. We consider in $\mathbb{R}^{d+1}$ with variables (x,t) $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$. With variables (ξ, τ) the characteristic equatopm $\Lambda(\xi,\tau) = -(\xi_1^2 + ... + \xi_d^2) = -|\xi|^2 = 0$. For example the vector (0,0,0,...,0,1) is a characteristic direction and the plane $\tau = 0$ is a characteristic surface. $F(x_1, ..., x_d, t) = t = 0$ $\nabla F = (0, ..., 0, 1)$

Example: Wave equation: $\frac{\partial^2 u}{\partial t^2} - \Delta u = f$. Consider in $\mathbb{R}^{d+1}$ with points (x,t), $x \in \mathbb{R}^d$, $t \in \mathbb{R}$. Characteristic equation with variables (ξ,τ), $\xi \in \mathbb{R}^d$, $\tau \in \mathbb{R}$. $\Lambda(\xi,\tau) = -(\xi_1^2 + ... + \xi_d^2) + \tau^2 = 0$, $\tau = \pm|\xi|$. Characteristic directions $(\xi, \pm|\xi|)$, $\xi \neq 0$ anything.

Characteristic surface: Given $\overline{x} \in \mathbb{R}^d$ and $\overline{t} \in \mathbb{R}$ consider the cone $|x - \overline{x}|^2 - |t - \overline{t}|^2 = 0$.
$\nabla F = (2(x_1 - \overline{x_1}), ..., 2(x_d - \overline{x_d}), -2(t - \overline{t})) = 2(x - \overline{x}, t - \overline{t}) \underset{t-\overline{t}=\pm|x-\overline{x}|}{=} 2(x - \overline{x}, \mp|x - \overline{x}|)$. This is of the form $(\xi, \pm|\xi|) \Rightarrow$ this cone is a characteristic surface.



Classification of 2:nd order PDE's:

Consider second order PDE with constant coefficients:

$$\sum_{j,k=1}^{d} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{d} b_j \frac{\partial u}{\partial x_j} + cu = f$$

where $a_{jk} = a_{kj}$, a_{jk}, b_j, c constants. Characteristic equation

$$\Lambda(\xi) = \sum_{j,k}^{d} a_{jk}\xi_j\xi_k = \xi \cdot A\xi \quad A = \begin{bmatrix} a_{11} & \dots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \dots & a_{dd} \end{bmatrix}$$

A is symmetric, we can use the *Spectral Theorem*

$$A = PDP^{-1},\ P^{-1} = P^T \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix}$$

We introduce a change of variables $P\eta = \xi$
$\Lambda(\xi) = \Lambda(P\eta) = P\eta \cdot AP\eta = P\eta \cdot PDP^{-1}P\eta = P\eta \cdot PD\eta = P^T P\eta \cdot D\eta = \eta D\eta = \sum_{j=1}^{d} \lambda_j \eta_j^2$.

Definition: A differential equation is elliptic if all λ_j has the same sign. It is hyperbolic if all but one λ_j has the same sign and it parabolic if the remaining $\lambda_j = 0$.

Let V be a vector space over $\mathbb{R}$.

Definition: An inner product on V is a function $V \times V \to \mathbb{R}$ such that

$$\begin{array}{ll} (1) & (\lambda u + \mu v, w) = \lambda(u, w) + \mu(v, w) \quad u, w \in V \ \lambda, \mu \in \mathbb{R} \\ (2) & (u, v) = (v, u) \quad u, v \in V \\ (3) & (v, v) > 0 \quad \text{for all } v \in V, v \neq 0 \end{array}$$

The pair $(V, (\cdot, \cdot))$ is called an inner product space.

Homework: Show that the following is true:

$$\begin{array}{ll} (a) & (v, v) = 0 \Leftrightarrow v = 0 \\ (b) & (w, \lambda u + \mu v) = \lambda(w, u) + \mu(w, v) \end{array}$$

Homework solution: This is shown by using our three axioms.

(a): We begin by showing $\Rightarrow$: Let $v = \lambda u$ where $u \neq 0$. Then it follows from axiom (1) that $(v, v) = (\lambda u, \lambda u) = \lambda(u, \lambda u)$. Then we use axiom (2) $\lambda(u, \lambda u) = \lambda(\lambda u, u) = \lambda^2(u, u)$. Since $(v, v) = 0$ it follows that $\lambda^2(u, u) = 0$ but we defined that $u \neq 0$ thus it follows from axiom (3) that $(u, u) > 0$ which means that $\lambda^2 = 0$, which implies that $v = 0$.

Now we show $\Leftarrow$: As before let $v = \lambda u$ where $u \neq 0$. By the same reasoning as before we have that $(v, v) = \lambda^2(u, u)$, and that $(u, u) > 0$. But since $v = 0$ and $u \neq 0$, λ has to be 0, which in turn means that $(v, v) = 0$.

(b): Axiom (2) gives us that $(w, \lambda u + \mu v) = (\lambda u + \mu v, w)$, axiom (1) then gives us that $(\lambda u + \mu v, w) = \lambda(u, w) + \mu(v, w)$. Finally we use axiom (2) again and we recieve $\lambda(u, w) + \mu(v, w) = \lambda(w, u) + \mu(w, v)$. $\square$

Example: Let $C[a, b]$ denote the set of real-valued continous functions on $[a, b]$ with addition $(f + g)(x) = f(x) + g(x)$ and scalar multiplication $(\lambda f)(x) = \lambda f(x)$. Define $(f, g) = \int_a^b f(x)g(x)dx$.

Homework: Show that $(C[a, b], (\cdot, \cdot))$ is an inner product space.

Homework solution: We have to show that the three axioms hold for all the elements in $C[a, b]$ with the given inner product.

(1): Consider $(\lambda f + \mu g, h)$, where f, g, h are arbitrary elements in $C[a, b]$ and λ, μ are arbitrary real constants. Our inner product gives us $\int_a^b (\lambda f(x) + \mu g(x))h(x)dx$, we use the linearity of the integral $\int_a^b (\lambda f(x) + \mu g(x))h(x)dx = \lambda \int_a^b f(x)h(x)dx + \mu \int_a^b g(x)h(x)dx$. Thus axiom (1) holds.

(2): Consider f, g defined as before. According to our inner product $(f, g) = \int_a^b f(x)g(x)dx = \int_a^b g(x)f(x)dx = (g, f)$. This means that axiom (2) holds.

(3): Consider $f \in C[a, b]$ such that f isn't the zero function on our interval. We have that $(f, f) = \int_a^b f(x)^2 dx$. $f(x)^2 \geq 0$ for all x and since it isn't the zero function $f(x)$ has to non-zero somewhere, thus $f(x)^2 > 0$ somewhere. Since we consider $f \in C[a, b]$ $f(x)^2$ has to be non-zero on atleast some interval in $[a, b]$ and 0 at least zero everywhere else, thus by the definition of the integral $\int_a^b f(x)^2 dx > 0 \Rightarrow (f, f) > 0$. Axiom (3) holds. $\square$

2 Lecture 2017.10.31

Definition: A linear functional is a function $f : V \to \mathbb{R}$ that is linear
$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$, $\lambda, \mu \in \mathbb{R}$ $u, v \in V$.

Definition: A bilinear form $a : V \times V \to \mathbb{R}$ is a function such that
$a(\lambda u + \mu v, w) = \lambda a(u, w) + \mu a(v, w)$ and $a(w, \lambda u + \mu v) = \lambda a(w, u) + \mu a(w, v)$, $u, v, w \in V$
$\lambda, \mu \in \mathbb{R}$. It is symmetric if $a(u, v) = a(v, u)$ and it is positive definite if $a(v, v) > 0$ for all $v \in V$ such that $v \neq 0$.

Homework: Let $V = (C[a, b], (\cdot, \cdot))$ be an inner product space with the inner product $(f, g) = \int_a^b fgdx)$. Show the following:

(a): $F(v) = \int_a^b v(x)dx$ is a linear functional.

(b): $F(v) = v(a)$ is a linear functional.

(c): $a(f, g) = \int_a^b f(x)g(x)(1 + x^2)dx$ is a positive definite bilinear form.

Homework solution: We use the definitions:

(a): Let v, u be elements from $C[a, b]$ and λ, μ elements from $\mathbb{R}$. Now consider
$F(\lambda v + \mu u) = \int_a^b \lambda u(x) + \mu v(x)dx = \lambda \int_a^b v(x)dx + \mu \int_a^b u(x)dx$. The integrals evaluate to real numbers. This mapping fulfills the condition defined above, it is linear in its argument and it maps functions to real numbers.

(b): Let u, v and λ, μ be defined as above. Now consider
$F(\lambda v + \mu u) = (\lambda v + \mu u)(a) = \lambda v(a) + \mu u(a)$. This mapping fulfills the condition defined above, it is linear in its argument and it maps functions to real numbers.

(c): Let $f, g, h \in C[a, b]$ and let $\lambda, \mu \in \mathbb{R}$. We begin by showing it's a bilinear form.
$a(\lambda f + \mu g, h) = \int_a^b (\lambda f(x) + \mu g(x))h(x)(1 + x^2)dx =$
$\lambda \int_a^b f(x)h(x)(1 + x^2)dx + \mu \int_a^b g(x)h(x)(1 + x^2)dx = \lambda a(f, h) + \mu a(g, h)$. We can see that if it is linear in its first argument a has to be linear in its second argument, following from elementary properties of the integral. To show that it is positive definite we consider
$a(f, f) = \int_a^b f(x)^2(1 + x^2)dx$ and let f not be the zero function. With $f \in C[a, b]$ we have that it has to be non-zero on atleast some interval in $[a, b]$, thus $f(x)^2$ is greater than zero on atleast some interval in $[a, b]$ and atleast zero everywhere else. Also, $(1 + x^2) > 0$ on $[a, b]$. Thus the integral has to be > 0, which means that a is positive definite. $\square$

Definition: We say that $u \in V$ and $v \in V$ are orthogonal if $(u, v) = 0$. Notation: $u \perp v$.

Definition: Let V be a vector space over $\mathbb{R}$ then a function $|| \cdot || : V \to \mathbb{R}_+$ is a norm on V if:

$$\begin{aligned}
&(a) \quad ||v|| > 0 \quad \forall v \neq 0 \\
&(b) \quad ||\lambda v|| = |\lambda|||v|| \quad \forall v \in V,\ \lambda \in \mathbb{R} \\
&(c) \quad ||u + v|| \leq ||u|| + ||v|| \quad u, v \in V
\end{aligned}$$

Note: $v = 0 \Leftrightarrow ||v|| = 0$. The pair $(v, || \cdot ||)$ is called a normed space.

Homework: Let $V = C[a, b]$ be a vector space with the norm $||f|| = \sup_{x \in [a,b]} |f| = \max_{x \in [a,b]} |f|$. Show that this is a normed space.

Homework solution: We have to show that the given norm fullfills the axioms given any element from V.

(a): $|f| \geq 0$, and since according to the axiom f can't be the zero function it has to be > 0

atleast on some interval. If we take the maximum value on that interval we will recieve a real number > 0.

(b): This follows directly from the properties of the supremum/maximum.
$\sup\limits_{x\in[a,b]} |\lambda f| = \lambda \sup\limits_{x\in[a,b]} |f|$.

(c): Let $f, g \in C[a,b]$ Consider $\sup\limits_{x\in[a,b]} |f+g|$ according to the triangle inequality for absolute values we have that $\sup\limits_{x\in[a,b]} |f+g| \leq \sup\limits_{x\in[a,b]} (|f|+|g|) \leq \sup\limits_{x\in[a,b]} |f| + \sup\limits_{x\in[a,b]} |g|$. Thus
$||f+g|| \leq ||f|| + ||g||$. □

If $(V, (\cdot,\cdot))$ is an inner product space then $||v|| = (v,v)^{1/2}$ is a norm.

Proposition: *Cauchy-Schwartz inequality*: Let $(V, (\cdot,\cdot))$ be an inner product space. Then $|(u,v)| \leq ||u||||v||$, $u, v \in V$ with equality if and only if $u = \lambda v$ for some $\lambda \in \mathbb{R}$.

Proof: If $v = 0$ the result holds trivially. Let $t \in \mathbb{R}$ and consider
$0 \leq (u+tv, u+tv) = ||u||^2 + 2t(u,v) + t^2||v||^2 := f(t)$. This is a quadratic function, since it's greater than 0 for all t it also has to be greater than 0 in its minimum. It can easily be shown that the minimum is $a = -\frac{(u,v)}{||v||^2}$.

$$0 \leq f(a) = ||u||^2 - 2\frac{(u,v)^2}{||v||^2} + \frac{(u,v)^2||v||^2}{||v||^4} = ||u||^2 - \frac{(u,v)^2}{||v||^2} \Rightarrow (u,v)^2 \leq ||u||^2||v||^2 \Rightarrow |(u,v)| \leq ||u||||v||$$

If $u = -tv$ we have equality. □

Proposition *Triangle inequality*: $||u+v|| \leq ||u|| + ||v||$.

Proof: We prove this by using *Cauchy-Schwartz inequality*

$$||u+v||^2 = (u+v, u+v) = ||u||^2 + 2(u,v) + ||v||^2 \leq ||u||^2 + 2||u||||v|| + ||v||^2$$
$$= (||u|| + ||v||)^2 \Rightarrow ||u+v|| \leq ||u|| + ||v||$$

□

Homework: Prove the *Parallellogram identity*: $||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$

Homework solution: We simply use the axioms and the definition of the norm!

$$||u+v||^2 + ||u-v||^2 = (u+v, u+v) + (u-v, u-v) = (u, u+v) + (v, u+v) + (u, u-v) - (v, u-v) =$$
$$(u,u) + (u,v) + (v,u) + (v,v) + (u,u) - (u,v) - (v,u) + (v,v) = 2(||u||^2 + ||v||^2) \qquad \square$$

Definition: Let $(x_n) \subset V$ be a sequence in $(V, ||\cdot||)$, we say $x_n \to x \in V$ as $n \to \infty$ alternatively written as $\lim\limits_{n\to\infty} x_n = x$ if $\lim\limits_{n\to\infty} ||x_n - x|| = 0$, with $\varepsilon - \delta$-notaion:
$(\forall\varepsilon > 0)(\exists N) : n \geq N \Rightarrow ||x_n - x|| < \varepsilon$.

Definition: A sequence is a *Cauchy-sequence* if $(\forall\varepsilon > 0)(\exists N) : m, n \geq N \Rightarrow ||x_n - x_m|| < \varepsilon$. It can be stated informally as: $\lim\limits_{m,n\to\infty} ||x_n - x_m|| = 0$.

Fact: If (x_n) is convergent then x_n is a Cauchy-sequence. ☠ The converse is not true! ☠

A normed space is called complete if every Cauchy-sequence converges. A complete normed space is called a *Banach space* and a complete inner product space is called a *Hilbert space*.

Example: $C[a,b]$, $||f|| = \sup\limits_{x\in[a,b]} |f|$ is a Banach space.

Homework: Show that $C[a,b]$, $||f|| = |\int_a^b f(x)^2|^{1/2}$ is not complete.

Homework solution:

Find a function that is Cauchy but that doesn't converge to a continous function. Try a function which converges to a step function.

Example:

$$V = \{(x_n)\}, \quad x_n \in \mathbb{R}, \quad \sum_{n=1}^{\infty} |x_n|^2 < \infty, \quad ((x_n),(y_n)) = \sum_{n=1}^{\infty} x_n \cdot y_n$$

$(V, (\cdot,\cdot))$ is complete.

Definition: Let V, W be normed spaces. A mapping $B : V \to W$ is linear if $B(\lambda u + \mu v) = \lambda Bu + \mu Bv \quad u, v \in V \quad \lambda, \mu \in \mathbb{R}$. It is bounded if there is $c > 0$ such that $||Bv||_W \leq c||v||_V$ for all $v \in V$. We nay then define the norm of B by

$$||B|| = \sup_{v\in V,\ v\neq 0} \frac{||Bv||_W}{||v||_V} = \sup_{||v||_V=1} ||Bv||_W = \inf\{c \in \mathbb{R} : ||Bv||_W \leq c||v||_V \text{for all } v \in V\}$$

$$\Rightarrow ||Bv||_W \leq ||B|| \cdot ||v||_V$$

Homework: Show the equalities above.

Homework solution:

Definition: We denote the set of bounded linear operators by $\mathcal{B}(V,W)$ if $V = W$, $\mathcal{B}(V)$. This can be made to be a vector space:

$$(B_1 + B_2)v = B_1 v + B_2 v \quad v \in V$$

$$(\lambda B)v = \lambda Bv \quad \lambda \in \mathbb{R},\ v \in V$$

Then $\mathcal{B}(V,W)$ is a normed space and if W is complete so is $\mathcal{B}(V,W)$.

Homework: Show that $||B||$ defined as above is a norm.

Homework solution:

Lemma: $B \in \mathcal{B}(V,W) \Leftrightarrow B$ is continous that is $x_n \to x \Rightarrow Bx_n \to Bx$.

Definition: Let V be a normed space. The space of continous linear functionals is $\mathcal{B}(V,\mathbb{R})$. Notation: $V^* = \mathcal{B}(V,\mathbb{R})$, V^* is called the dual space of V. Since $\mathbb{R}$ is complete so is V^*.

A bilinear form $a : V \times V \to \mathbb{R}$ is bounded if there is $c > 0$ sicj that $|a(u,v)| \leq c||u|| \cdot ||v||$.

Definition: The ball centerad at $v_0 \in V$ with radius $r > 0$ is $B_r(r_0) = \{v \in V \ ||v - v_0|| < r\}$.

Definition: A set $A \subset V$ is open if for every $v_0 \in A$ there is $r = r(v_0)$ such that $B_r(v_0) \subset A$.

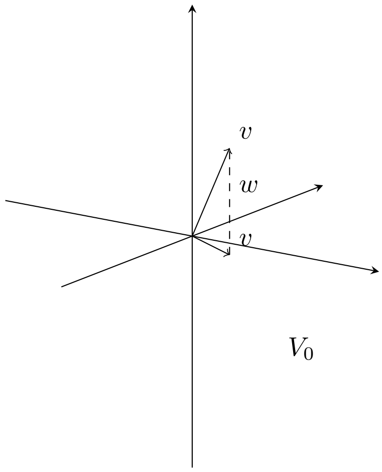
Definition: A is closed if $A^c = V \setminus A$ is open.

Homework: Show that A is closed $\Leftrightarrow (x_n) \in A$, $x_n \to x \in V \Rightarrow x \in A$.

Homework solution:

Definition: $A \in V$ is a dense subset of V of for all $v \in V$ there is $v_n \in A$ $v_n \to v$.

Theorem: Let V be a Hilbert space and $V_0 \subset V$ be a closed subspace. Then any $v \in V$ can be uniquely be written as $v = v_0 + w$ where $v_0 \in V_0$ and $w \perp v_0$. The element v_0 can be characterised as th unique element in V_0 such that $||v - v_0|| = \min\{||v - u||, u \in V_0\}$. The element v_0 is denoted by $P_{V_0}v$.



3 Lecture 2017.11.06

Corollary: V is a Hilbert space, $V_0 \subset V$ is a closed subspace, $V_0 \neq V$. Then $w \in V \setminus V_0$, $w \perp v_0$

Proposition: $V_0 \neq V \Rightarrow \exists w_0 \in V \setminus V_0, \quad w_0 \neq 0$. Projection theorem:
$w_0 = v_0 + w, \quad w \perp v_0 \quad w \neq 0$ as $w_0 \neq v_0$.

Theorem: (*Riesz Representation Theorem*) Let V be a Hilbert space and $L : V \to \mathbb{R}$ be a bounded linear functional on V (ie. $L \in V^*$). Then there is a unique $u \in V$ such that $L(V) = (v, u)$ for all $v \in V$. Furthermore $||L||_V = ||u||$.

Proof: See the book.

Note: The Riesz representation theorem identifies continous linear functionals with elements of the Hilbert space V.

Homework: Show that the map $\Phi : L \to u$ $(V^* \to V)$ is linear, surjective and isometric. (V and V^* are isometrically isomorphic).

Homework solution:

Often in this course we will study the following problem: Let V be a Hilbert space and $L : V \to \mathbb{R}$ be a bounded and $a : V \times V \to \mathbb{R}$ bilinear positive definite. Problem: Find $u \in V$ such that $a(u, v) = L(v)$ for all $v \in V$. Call this problem (V).

Definition: A bilinear form $u : V \times V \to R$ is called coercive of there is an $\alpha > 0$ sich that $a(v, v) \geq \alpha ||v||^2$ for all $v \in V$. Note that coercive implies positive definite, but positive definite does not imply coercive. In finite dimensions however, postitive definite and coercive is equivalent.

If $a : V \times V \to \mathbb{R}$ is positive definite, symmetric and bilinear, then a is an inner product on V.

If a is coercive and bounded, then the norm (energy norm) $||v||_a = a(v, v)^{1/2}$ is equivalent to the original norm $|| \cdot ||$. $\alpha ||v||^2 \leq a(v, v) \leq M ||v||^2$.

In summary: If $a : V \times V \to \mathbb{R}$ is bilinear, coercive, symmetric and bounded then: the energy norm $|| \cdot ||_a$ and $|| \cdot ||$ are equivalent and therefore $(V, || \cdot ||_a)$ is complete (hence a Hilbert space). Also L is bounded linear on $(V, || \cdot || \Rightarrow)$ bounded linear on $(V, || \cdot ||_a)$.

In this case the Riesz representation theorem on $(V, || \cdot ||_a)$ yields that there is an unique $u \in V : L(v) = a(v, u) = a(u, v)$ for all $v \in V$. Thus equation (V) has a unique solution.

Energy estimate: We may bound the norm of the solution in terms of L:
$\alpha ||u||^2 \leq a(u, u) = L(u) \leq ||L||_{V^*} ||u||_V \Rightarrow ||u||_V \leq \frac{1}{\alpha} ||L||_{V^*}$.

The solution to (V) may be characterized through a minimization problem:

Theorem: If $a : V \times V \to \mathbb{R}$ is symmetric and positive definite then u is a solution to problem $(V) \Leftrightarrow F(u) \leq F(v)$ for all $v \in V$ $F(u)) = \frac{1}{2} a(u, u) - L(u)$

Proof: Suppose that u is a solution to (V). Set $w = v - u \Rightarrow v = u + w$. Then

$$F(v) = F(u + w) = \frac{1}{2} a(u + w, u + w) - L(u + w) = \frac{1}{2} a(u, u) - L(u) + a(u, w) - L(w) + \frac{1}{2} a(w, w)$$

The sum of the first two terms are equal to $F(u)$ by definition. The som of the second two terms are equal to 0 since u is a solution. Thus we have $F(v) \geq F(u)$ since $a(w, w) \geq 0$.

Now suppose $F(u) \leq F(v)$ for all $v \in V$. Consider $g(t) = F(u + tv) \geq F(u) = g(0)$, where t is a real parameter. we have

$$g(t) = F(u+tv) = \frac{1}{2}a(u+tv, u+tv) - L(u+tv) = \frac{1}{2}t^2 a(v,v) + (a(u,v) - L(v))t + \frac{1}{2}a(u,u) - L(u)$$

This is a quadratic in t and it has a minimum at 0 thus
$0 = g'(0) = a(u,v) - L(v) \Rightarrow a(u,v) = L(u)$ $\square$

Note: F is called the energy functional and (V) the variational equation for F.

There is an extension when a is non-symmetric.

Theorem: (Lax-Milgram) Let V be a Hilbert space and $a : V \times V \to \mathbb{R}$ be a bounded coercive bilinear form and $L : V \to \mathbb{R}$ be a bounded linear functional then there is a unique $u \in V$ sich that $a(u,v) = L(v)$ for all $v \in V$. (That is (V) has a unique solution)

Note: Unlike the symmetric case before there is no characterization of u through the minimization of an energy functional. But we still have $||u|| \leq \frac{1}{\alpha}||L||_{V^*}$.

Function spaces: Let $\Omega \subset \mathbb{R}^d$ then $\overline{\Omega}$ denotes the closure of Ω.

$$\overline{\Omega} = \bigcap_{\Omega \subset A, A \text{ is closed}} A$$

An example is that the closure of a ball is the ball with its boundary.

Let Ω be a domain ($\equiv$ open, connected). $C(\Omega)$: vector space of continuous functions $\Omega \to \mathbb{R}$.

If Ω is abounded domain then $C(\overline{\Omega})$ is a Banach space with norm
$||V||_{C(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} |v(x)| = \max_{x \in \overline{\Omega}} |v(x)|$

$C^k(\Omega)$: space of k-times continually differentiable functions on Ω : then $D^\alpha v$ is continous for all $|\alpha| \leq k$.

$C^k(\Omega) : \{v \in C^k(\Omega) : D^\alpha v \in C(\overline{\Omega}), |\alpha| \leq k\}$. This is a Banach space if we set
$||v||_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq k} ||D^\alpha_v||_{C(\overline{\Omega})}$. In 1D: $\Omega = (0,1)$:

$$||v||_{C^2(\Omega)} = \sup_{x \in [0,1]} |v(x)| + \sup_{x \in [0,1]} |v'(x)| + \sup_{x \in [0,1]} |v''(x)|$$

A function $V : \Omega \to \mathbb{R}$ has compact support if $v = 0$ outside of a compact set (compact $\Leftrightarrow$ bounded and closed in $\mathbb{R}^d$)

$C^k_0(\Omega)$ is the space of functions in $C^k(\Omega)$ with compact support.

$C^\infty_0(\Omega)$: $v \in C^k_0(\Omega)$ for every k.

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Definition: Let $\Omega \subset \mathbb{R}^d$ be a domain. To begin with let, $1 \leq p < \infty$. A function $v \in L^p(\Omega)$ if $\int_\Omega |v(x)|^p dx < \infty$. We define $||v||_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p dx\right)^{1/p}$. Here follows a couple of notes regarding this definition.

Note 1: Here $\int_\Omega f(x)dx$ denotes the *Lebesgue* integral. It coincides with the Riemann integral for bounded Riemann integrable functions (at least on bounded Ω). For such functions the Lebesgue integral is an extension of the Riemann integral.

Note 2: There are many functions that are not Riemann integrable but are Lebesgue integegrable.

Example: $\Omega = (0,1)$, consider the Dirichlet-function:

$$v(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Note that v is very simple $v = \chi_{\mathbb{Q}\cap(0,1)}$. It's easy to see that v is not Riemann integrable, however it is Lebesgue integrable and $\int_\Omega v(x)dx = 0$.

Note 3: The Lebesgue integral behaves much nicer than the Riemann integral if one wants to exchange limits and integrals.

Example: Suppose $f_n(x) \to f(x)$, $f \in \Omega$. Then $||f_n(x)| \leq g(x)$, $g(x) \in L^1(\Omega) \Rightarrow$ $\int_\Omega f(x)dx = \lim_{n\to\infty} \int_{Omega} fn(x)dx$. This is called Lebesgue's dominated convergence theorem.

Note 4: We consider two functions v and w equivalent, or we say that they are equal almost everywhere (a.e) if $v(x) \neq w(x)$ only for $x \in A$ where A has Lebesgue measure 0, defined as follows: Let $c = (a_1, b_1) \times ... \times (a_d, b_d) \subset \mathbb{R}^d$ be a hypercube in $\mathbb{R}^d$. The Lebesgue measure $m(c)$ of c is defined by $m(c) = \prod_{i=1}^d (b_i - a_i)$.

Definition: A set $A \subset \mathbb{R}^d$ has Lebesgue measure 0 if for every $\varepsilon > 0$ there are countably many hypercubes $c_n,\ n = 1, 2, ...$ such that $A \subset \bigcup_{n=1}^\infty c_n$ and $\sum_{n=1}^\infty m(c_n) < \varepsilon$. Note that if $A = \{a\} \Rightarrow m(A) = 0$, if A is countable then $m(A) = 0$.

Example: Consider $\mathbb{R}^2$ then the real line $A = \{(x, 0), x \in \mathbb{R}\}$ has Lebesgue measure 0 (a line has 0 "area"). In general if $\Omega \subset \mathbb{R}^d$ a domain, then the boundary Γ of Ω ($\Gamma = \overline{\Omega} \setminus \Omega$) has Lebesgue measure 0. for example $\{(x, 0), x \in \mathbb{R}\} = \Gamma,\ \Omega = \{(x, y) : x \in \mathbb{R}\ y > 0\}$.

Note 5: If $v = w$ a.e, then if v is Lebesgue integrable then so is w and $\int_\Omega vdx = \int_\Omega wdx$.

Example: With the Dirichlet-function from before $v \equiv 0$ a.e because $m(\mathbb{Q} \cap (0,1)) = 0$ thus v is Lebesgue integrable with Lebesgue integral 0.

Note 6: Elements of the space $L^p(\Omega)$ are equivalence classes of functions that are equal a.e. Therefore in general we cannot talk about point values of $v \in L^p(\Omega)$, that is $v(x)$ for fixed x (unless there is a continous representation in the equivalence class).

Note 7: $L^p(\Omega)$ is complete and hence a Banach space. $p = 2,\ L^2(\Omega)$ is a Hilbert space with inner product $(u, v) = \int_\Omega uvdx$ where this is the Lebesgue integral.

Note 8: Regarding $p = \infty$. We say that v is essentially bounded if there is a $M > 0$ such that $|v(x)| \leq M$ for almost all x.

$$||v||_{L^\infty} = \inf\{M : |v(x)| \leq M \text{ for almost all } x\} \overset{\text{def}}{=} \operatorname*{ess\,sup}_{x\in\Omega} |v(x)| \neq \sup_{x\in\Omega} |v(x)|$$

L^∞ is a Banach space.

Example: $\Omega = (0,1)$ and for $n = 1, 2, ...$

$$\begin{cases} 1 & \text{if } x \neq \frac{1}{n} \\ n & \text{if } x = \frac{1}{n} \end{cases}$$

$\sup_{x\in\Omega} |v(x)| = \infty$ but $\operatorname*{ess\,sup}_{x\in\Omega} |v(x)| = 1$.

Note 9: If the boundary Γ of Ω is smooth enough (say, Lipschitz continous) then $C_0^k(\Omega)$ (also $C_0^\infty(\Omega)$) is dense in $L^p(\Omega)$, $1 \leq p < \infty$. That is for every $v \in L^p(\Omega)$ there are $(v_n) \subset C_0^k\Omega$ (resp $C_0^\infty(\Omega)$) such that $||v_n - v||_{L^p} \to 0$ as $n \to \infty$. This does not hold for L^∞.

Sobolov spaces: We need the concept of weak (or generalized or distrubutional) derivatives. We begin with a lemma.

Lemma: Suppose that V and W are Banach spaces and $A \subset V$ is a dense subspace of V (dense: $\forall v \in V \exists (v_n) \subset A : v_n \to v$). Supposse that $B : A \to W$ is a bounded linear operator. Then there is a unique linear continuous ($\equiv$ bounded) extension $\tilde{B}$ of B to the the whole of V such that $||\tilde{B}||_{\mathcal{B}(V,W)} = ||B||_{\mathcal{B}(A,W)}$.

Let $\Omega \subset \mathbb{R}^d$ be a domain. Let $v \in C^1(\overline{\Omega})$. Let $\Phi \in C_0^1(\Omega)$. Integrate by parts:

$$(*) = \int_\Omega \frac{\partial v}{\partial x_i} \Phi dx = -\int_\Omega v \frac{\partial \Phi}{\partial x_i} dx$$

This is a special case of Greens formula (see introduction of the book) $w = (w_1, ..., w_d)$ vector field, ψ scalar field then

$$\int_\Omega w \cdot \nabla \psi dx = \int_\Gamma w \cdot n \psi dx - \int_\Omega \nabla w \psi dx$$

n is the outward facing unit normal of Γ.

If $v \in L^2(\Omega)$ it might not have a classical derivative. One can define the generalized (weak) derivative denoted by $\frac{\partial v}{\partial x_i}$ to be a functional with the following properties:

Definition: The weak derivative is defined as

$$\frac{\partial v}{\partial x_i}(\Phi) = L(\Phi) = -\int_\Omega v \frac{\partial \Phi}{\partial x_i} dx, \;\; \Phi \in C_0^1(\Omega)$$

Suppose that L is bounded that is there is a $M > 0$ such that $|L(\Phi)| \leq M ||\Phi||_{L^2} \; \forall \Phi \in C_0^1(\Omega)$. Then by the lemma ther is a continous linear extension of L to the whole of L^2 (because C_0^1 is dense in L^2). By Riesz representation theorem there is an unique $w \in L^2$ such that $L(\Phi) = (\Phi, w)$ $\Phi \in L^2$. Therefore in this case

$$\int_\Omega v \frac{\partial \Phi}{\partial x_i} dx = L(\Phi) = \int_\Omega \Phi w dx \; \forall \Phi \in C_0^1$$

In this case we say that $\frac{\partial v}{\partial x_i}$ is in L^2. We still denote w by $\frac{\partial v}{\partial x_i}$. With this notation

$$(**) = -\int_\Omega v \frac{\partial \Phi}{\partial x_i} dx = \int_\Omega \Phi \frac{\partial v}{\partial x_i}, \;\; \forall \Phi \in C_0^1(\Omega)$$

Comparing $(*)$ with $(**)$ we say that for $v \in C_0^1(\overline{\Omega})$ the weak derivative coincides with the classical derivative. Note: weak derivative allows for integration by parts in the appropriate way.

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Let α be a multiindex and $v \in L^2(\Omega)$. Define $D^\alpha v$ as a functional:

$$(D^\alpha v)(\Phi) = L(\Phi) = (-1^{|\alpha|} \int_\Omega) v D^\alpha \Phi dx, \quad \Phi \in C_0^{|\alpha|}(\Omega)$$

If $|L(\Phi)| \leq ; ||\Phi||_{L^2}$ then since $\Phi \in C_0^{|\alpha|}(\Omega)$ is dense, there is a unique continuous extension of L to the whole of L^2. By the Riesz representation theorem there is $w \in L^2$ which we denote by $D^\alpha v$ such that $(w, \phi) = (D^\alpha v, \Phi) = L(\Phi) = (-1)^{|\alpha|} \int_\Omega v D^\alpha \Phi dx = (-1)^{|\alpha|}(v, D^\alpha \Phi)$, $\forall \Phi \in C_0^{|\alpha|}(\Omega)$.

Definition: The *Sobolev space* $H^k(\Omega)$ is defined by:

$$H^k(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega) \ |\alpha| \leq k\}$$

We endow H^k with the inner product

$$(u, v)_{H^k} = (u, v)_k = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v dx$$

and with the norm:

$$||u||_{H^k} = ||u||_k = \left(\sum_{|\alpha| \leq k} \int_\Omega (D^\alpha u)^2 dx \right)^{1/2}$$

Note: For H^0 we have $||v||_0 = ||v||_{H^0} = ||v||_{L^2} = ||v||$. For H^1 we have:

$$||v||_1 = \left(\int_\Omega v^2 + \sum_{j=1}^d (\frac{\partial v}{\partial x_j})^2 dx \right)^{1/2}$$

and for H^2 we have:

$$||v||_2 = \left(\int_\Omega v^2 + \sum_{j=1}^d (\frac{\partial v}{\partial x_j})^2 + \sum_{j=1}^d \sum_{k=1}^d (\frac{\partial^2 v}{\partial x_j \partial x_k}) dx \right)^{1/2}$$

note that the H^2 norm contains all the mixed second order derivatives not just the Laplacian! We continue by listing two important properties of the Sobolev spaces.

Property 1: H^k is a Hilbert space

Property 2: $C^l(\overline{\Omega})$ is a dense subspace of $H^k(\Omega)$ for $l \geq k$, this holds if $\Gamma = \partial\Omega$ is smooth enough.

Definition: The seminorm $|\cdot|_k$ is defined by:

$$|v|_k = \left(\sum_{|\alpha| = k} \int_\Omega (D^\alpha v)^2 dx \right)^{1/2}$$

This is not a norm, for example $|v|_k = 0$ for $v =$ constant. Still the triangle inequality holds and $|\lambda v|_k = \lambda |v|_k$.

Definition: We define the trace. This is the generalization of the boundary value of a function. If $v \in C^k(\overline{\Omega})$ then we may define the boundary value γv of v by restricting v to $\Gamma : (\gamma v)(x) = v(x) \ x \in \Gamma$. Then γv is a continuous function on Γ. We would like to extend this concept to $v \in H^1$.

Problem: Γ has the Lebesgue measure 0 in $\mathbb{R}^d$. As functions in H^1 are only defined as L^2 functions the point values on Γ are not well defined.

Idea: We define the boundary space $L^2(\Gamma)$ as the space of functions on Γ such that the surface integral $\int_\Gamma v^2 ds < \infty$, with the norm $||v||_{L^2(\Gamma)} = \left(\int_\Gamma v^2 ds\right)^{1/2}$. We will first define the boundary value of a function $v \in C^1(\Omega) \subset H^1$ by restriction of v to the boundary and we try to extend this notion to the whole of H^1 using the denseness of $C^1(\Omega)$ in H^1.

Lemma: Let $\Omega = (0,1)$. Then there is a constant $c > 0$ sich that $|v(x)| \leq C||v||_1$ for all $c \in C^1(\overline{\Omega})$ and $x \in \overline{\Omega}$ (in particular we may take $x = 0, 1$).

Proof: For $x, y \in \Omega$ and $v \in C^1(\overline{\Omega})$ we have $v(x) = v(y) + \int_y^x v'(s)ds$ (this is nothing but usage of the fundamental theorem of integral calculus). Then we use the triangle inequality, the triangle inequality for integrals and Cauchy-Schwarz

$$|v(x) \leq |v(y)| + |\int_y^x 1 \cdot v'(s)ds| \leq |v(y)| + \int_y^x 1 \cdot |v'(s)|ds \leq |v(y)| + \left(\int_0^1 1^2 ds\right)^{1/2} \left(\int_0^1 |v'(s)|^2 ds\right)$$

The limits of integration can change from x, y to $0, 1$ since the absolute value makes the integral grow when the interval grows, thus it is fine to make enlarge our limits to the whole of Ω in our inequality. Then we use $(a+b)^2 \leq 2a^2 + 2b^2$:

$$|v(x)|^2 \leq 2\left(|v(y)|^2 + \int_0^1 |v'(s)|^2 ds\right)$$

Since the righthand side is independent of y and the second term on the lefthand side is independent of y we can take the integral with respect to y on both sides (since the length of our integral is 1 these objects integrate like multiplication with 1) and acquire

$$|v(x)|^2 \leq 2\left(||v||_{L^2}^2 + ||v'||_{L^2}^2\right) = 2||v||_1^2$$

By continuity this result holds for $x \in \overline{\Omega}$. We have $|v(1)| = \lim_{n \to 1} |v(x)|$ and $x_n \to x,\ |x_n| \leq m \Rightarrow |x| \leq m$. This concludes the proof. □

Theorem: (Trace theorem) Let $\Omega \in \mathbb{R}^d$ be a bounded domain. Suppose that $\Gamma = \partial\Omega$ is a polygon or smooth. We define the trace operator γ by $\gamma : C^1(\overline{\Omega}) \subset H^1(\Omega) \to C^1(\Gamma) \subset L^2(\Gamma)$ $(\partial v)(x) = v(x)\ x \in \Gamma$. Then there is a bounded linear extension of γ to the whole of $H^1(\Omega)$ still denoted by γ. In particular there is a $c > 0$ sich athat $||\gamma v||_{L^2(\Gamma)} \leq c||v||_{H^1(\Omega)}\ \forall v \in H^1(\Omega)$.

Note: In this settomg the "boundary value" of a function in $H^1(\Omega)$ only exists as a function on $L^2(\Gamma)$.

Proof: γ is clearly linear. By homework problem 2.5 we only need to show that $||\gamma v||_{L^2(\Gamma)} \leq c||v||_{H^1}\ v \in C^1(\overline{\Omega})$ as $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$. We will prove this for $(0,1) \times (0,1)$ We will only consider one side of the rectangle, the same reasoning as follows holds for the other three. Let $(x_1, x_2) \in \Omega$ we use the lemma applied to the function $x \to v(x_1, x_2)$ and $x_1 = 0$ (right side of the rectangle).

$$v(0, x_2)^2 \leq 2\left(\int_0^1 v(x_1, x_2)^2 dx_1 + \int_0^1 (\frac{\partial v(x_1, x_2)}{\partial x_1})^2 dx_1\right)$$

$$\int_0^1 v(0, x_2)^2 dx_2 \leq 2\left(\int_0^1 \int_0^1 v(x_1, x_2)^2 dx_1 dx_2 + \int_0^1 \int_0^1 (\frac{\partial v(x_1, x_2)}{\partial x_1})^2 dx_1 dx_2\right) \leq 2\left(||v||_{L^2(\Omega)}^2 + ||\nabla v||_{L^2(\Omega)}^2\right)$$

This implies that $||v||_{L^2(\Gamma)} \leq 2||v||_1^2$ □

Definition: We saw that the trace operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is bounded and therfore it's nullspace (kernel) is a closed subspace of H^1_Ω. We define H^1_0:

$$H^1_0(\Omega) = \{v \in H^1(\Omega)_\gamma v = 0\}$$

It is a closed subspace of H^1 these are all the functions in H^1 that vanish on the boundary Γ in the trace sense.

Homework: $T : V \to W$, where V and W are normed spaces, is bounded. Show that $\ker(T) = \{v \in V : Tv = 0\}$ is a closed subspace of V.

Homework solution:

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Theorem:(Poincaré inequality) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Then there is a constant c such that $||v||_{L^2} \le c||\nabla v||_{L^2}$ for all $v \in H_0^1$. It is important that $v \in H_0^1$ (zero on boundary).

Proof: Fact: C_0^1 is dense in H_0^1 therefore it is enough to prove that $||v||_{L^2} \le c||\nabla v||_{L^2}$ $\forall v \in C_0^1(\Omega)$. Indeed: $v \in H_0^1, \exists (v_n) \in C_0^1 : v_n \to v$ in H^1-norm $v_n \to v$ in L^2, $\nabla v_n \to \nabla v$ in $L^2 \Rightarrow$

$$||v_n||_{L^2} \le c||\nabla v_n||_{L^2} \longrightarrow ||v||_{L^2} \le ||\nabla v||_{L^2} \text{ as } n \to \infty$$

as $|| \cdot ||_{L^2}$ is continuous. We will prove this for $\Omega = (0,1) \times (0,1)$. Let $v \in C_0^1(\Omega)$ $x \in (x_1, x_2) \in \Omega$. Then:

$$v(x_1, x_2) - v(0, x_2) = \int_0^{x_1} \frac{\partial v}{\partial x_1}(s, x_2) ds$$

This is simply the fundamental theorem of calculus. The second term on the righthand side is 0 because of compact support. We now use Cauchy-Schwarz, our second facor is the invisible 1 in front of our derivative of v:

$$v(x_1, x_2)^2 \le \int_0^{x_1} 1^2 ds \cdot \int_0^{x_1} \left(\frac{\partial v}{\partial x_1}(s, x_2) \right)^2 ds \le \int_0^1 \left(\frac{\partial v}{\partial x_1}(s, x_2) \right)^2 ds.$$

Here the last inequality follows from $x_1 \le 1$, since we have a squared realvalued function the integral can only get bigger if we extend our integration limits. We now integrate the above inequality over all of Ω:

$$\int_0^1 \int_0^1 v(x_1, x_2)^2 dx_1 dx_2 \le \int_0^1 \int_0^1 \left(\frac{\partial v}{\partial x_1}(s, x_2) \right)^2 ds dx_2.$$

The integral over x_1 on the righthand side evaluates to 1 since the righthand side doesn't depend on x_1. The righthand side definitely is smaller than the norm of the gradient squared, if we add more derivative terms we will end up with something larger. Thus we have:

$$\int_0^1 \int_0^1 v(x_1, x_2)^2 dx_1 dx_2 \le \int_0^1 \int_0^1 \left(\frac{\partial v}{\partial x_1}(s, x_2) \right)^2 ds dx_2 \le ||\nabla v||_{L^2}^2,$$

which we wanted to show. □

Corollary: If $v \in H_0^1$ then:

$$|v|_1^2 = ||\nabla v||_{L^2}^2 \le ||v||_{L^2}^2 + ||\nabla v||_{L^2}^2 (= ||v||_1^2) \le c||\nabla v||_{L^2}^2 + ||\nabla v||_{L^2}^2 = (c+1)||\nabla v||_{L^2}^2 = (c+1)|v|_1^2.$$

Therefore on H_0^1 $| \cdot |_1$ and $|| \cdot ||_1$ are equivalent and thus $| \cdot |_1$ is a norm on H_0^1 not just a seminorm.

Definition: The dual space $(H_0^1)^*$ is denoted bu H^{-1}. That is H^{-1} is the space of bounded linear functionals on H_0^1. If we equip H_0^1 with $| \cdot |_1$ then the norm on H^{-1} is given by

$$||L||_{H^{-1}} = \sup_{v \in H_0^1} \frac{|L(v)|}{|v|_1}.$$

Boundary value problems: We will consider a general second order elliptic problem of the form (which we will refer to as BVP):

$$\mathcal{L}u = -\nabla \cdot (a \nabla u) + b \cdot \nabla u + cu = f$$

where $f \in \Omega \subset \mathbb{R}^d$ and $u = 0$ on Γ. a, b and c are smooth functions (b vectorfield) and f is continuous.

Definition: A function u is a classical solution of the boundary value problem if $u \in C^2\overline{\Omega}$ and u satisfies BVP.

Note: In applications one would like to consider more genreral f, say $f \in L^2$. We need a more general solution concept, weak or variational formulation of BVP.

Suppose that $u \in C^2(\overline{\Omega})$ is a classical solution. We take $v \in C_0^1(\Omega)$ multiply both sides of the equation BVP by v and integrate over Ω (note: integration by parts):

$$\int_\Omega fvdx = \int_\Omega \mathcal{L}uvdx = \int_\Omega -\nabla\cdot(a\nabla u)v+b\cdot\nabla uv+cuvdx = -\int_\Gamma a\nabla u\cdot nvds+\int_\Omega a\nabla u\cdot\nabla v+b\cdot\nabla uv+cuvdx.$$

The integral over Γ is 0 since $v \in C_0^1$. Thus we have we have:

$$\int_\Omega a\nabla u \cdot \nabla v + b \cdot \nabla uv + cuvdx = \int_\Omega fvdx \ \forall v \in C_0^1(\Omega)$$

Claim: This holds for all $v \in H_0^1(\Omega)$. $v \in H_0^1$, $(v_n) \in C_0^1$ such that $v_n \to v$ in L^2 and $\nabla v_n \to \nabla v$ in L^2. Thus our equation can be extended to H_0^1 by taking the limit $n \to \infty$, we also note that our integral is a sum of inner products in L^2:

$$(a\nabla u, v_n) + (b \cdot \nabla u, v_n) + (cu, v_n) = (f, v_n) \longrightarrow (a\nabla u, v) + (b \cdot \nabla u, v) + (cu, v) = (f, v).$$

Definition: (Weak/Variational solution of BVP) Find $u \in H_0^1$ such that

$$\int_\Omega a\nabla u \cdot \nabla v + b \cdot \nabla uv + cuvdx = \int_\Omega fvdx, \ \forall v \in H_0^1.$$

Terminology: Such a function u is called a weak or variational solution of BVP. Note: The above calculation shows that a classical solution is weak solution. Conversely: If u is a weak solution and $u \in C^2(\overline{\Omega})$ then u is a classical solution. Reversing the above calculation we find that

$$\int_\Omega fvdx = \int_\Omega \mathcal{L}uvdx \ \forall v \in C_0^1$$

or

$$\int_\Omega (\mathcal{L}u - f)vdx = 0 \ \forall v \in C_0^1$$

$(\mathcal{L}u - f, v) = 0 \ \forall v \in C_0^1$. As C_0^1 is dense in L^2 we conclude that $\mathcal{L}u - f = 0$ in L^2 that is $\mathcal{L}u - f = 0$ a.e. If $u \in C^2(\overline{\Omega})$ and $f \in C(\Omega) \Rightarrow \mathcal{L}u - f \in C(\Omega) \Rightarrow \mathcal{L}u(x) - f(x) = 0$ for all $x \in \Omega$. (If g is continuous on Ω and $g = 0$ a.e then $g = 0 \ \forall x \in \Omega$) Finally as $u \in H_0^1 \cap C^2(\overline{\Omega})$, we have $(\gamma u)(x) = u(x),\ x \in \Gamma \Rightarrow u = 0$ onΓ thus u is a classical solution.

Note: A weak solution is often not regular enough to be a classical solution (e.g $f \in L^2$, Ω has corners etc.).

Theorem: Suppose that a, b and c are smooth functions in $\overline{\Omega}$ and that $a(x) \geq a_0 > 0$ and that $c(x) - \frac{1}{2}\nabla \cdot b \geq 0$ for all $x \in \Omega$ and $f \in L^2$. Then there is a unique weak solution u of BVP. That is, there is a unique $u \in H_0^1$ such that

$$\int_\Omega a\nabla u \cdot \nabla v + b \cdot \nabla uv + cuvdx = \int_\Omega fvdx \ \forall v \in H_0^1.$$

Furthermore there is a constant $c > 0$ independent of f sich that $|u|_1 \leq c||f||_{L^2}$.

Proof: We will use the Lax-Milgram Lemma on $V = H_0^1$ with norm $|\cdot|_1$, bilinear form

$$a(w, v) = \int_\Omega a\nabla w \cdot \nabla v + b \cdot \nabla wv + cwvdx \ v, w \in H_0^1 = V$$

and linear functional $L(v) = \int_\Omega fvdx$. We need to check that a is bilinear bounded and coercive, we also need to check that $L : V \to \mathbb{R}$ is bounded.

To begin with we will need some inequalities they are

$||f \cdot g||_{L^2} \leq ||f||_{L^\infty} \cdot ||g||_{L^2}$

If $F = (f_1, ..., f_d)$ $G = (g_1, ..., g_d) | \int_\Omega F \cdot G dx| \leq ||F||_{L^2} \cdot ||G||_{L^2}$ where $||F||_{L^2} = \int_\Omega \sum_{j=1}^{d} f_j^2 dx$

$||F \cdot G||_{L^2} \leq \max_{1 \leq i \leq d} ||f_i||_{L^\infty} ||G||_{L^2}$

$||fF||_{L^2} \leq ||f||_{L^\infty} ||F||_{L^2}$.

The proof continues in the next lecture.

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Proof: We will use Lax-Milgram Lemma: If V is a Hilbert space, $a : V \times V \to \mathbb{R}$ is a bounded coercive bilinear form on V and $L : V \to \mathbb{R}$ is a bounded linear functional on V then there is a unique $u \in V$ such that $a(u, v) = L(v)\ \forall v \in V$ and $||u||_V \leq c||L||_{V^*} = \sup_{v \in V} \frac{|L(v)|}{||v||_V}$.

Let $V = H_0^1$ with norm $|\cdot|_1$, define

$$a(w, v) = \int_\Omega a\nabla w \cdot \nabla v + b \cdot \nabla wv + cwvdx\ v, w \in H_0^1 = V$$

and define

$$L(v) = \int_\Omega fvdx\ v \in H_0^1 = V.$$

As stated we need to show: a is (1) bilinear, (2) bounded and (3) coercive, we also have to check if (4) L is bounded. It is easy to see that a is bilinear, that takes care of criterion (1). We now show that a is bounded, that is $|a(w, v)| \leq K|w|_1|v|_1$:

$$|a(w, v)| \leq \left|\int_\Omega a\nabla w \cdot \nabla vdx\right| + \left|\int_\Omega b \cdot \nabla wvdx\right| + \left|\int_\Omega cwvdx\right| \overset{\text{C.S}}{\leq}$$

$$||a\nabla w||_{L^2}||\nabla v||_{L^2} + ||b \cdot \nabla w||_{L^2}||v||_{L^2} + ||cw||_{L^2}||v||_{L^2} \leq$$

$$||a||_{L^\infty}||\nabla w||_{L^2}||\nabla v||_{L^2} + \left(\max_{1\leq i\leq d}||b_i||_{L^\infty}\right)||\nabla w||_{L^2}||v||_{L^2} + ||c||_{L^\infty}||w||_{L^2}||v||_{L^2} \overset{\text{Poincaré}}{\leq}$$

$$||a||_{L^\infty}|v|_1|w|_1 + M\left(\max_{1\leq i\leq d}||b_i||_{L^\infty}\right)|w|_1|v|_1 + M^2||c||_{L^\infty}|v|_1|w|_1 \leq K|w|_1|v|_1$$

(note that we have used the definition of the seminorm here) where

$$K = 3\max\left\{||a||_{L^\infty}, +M\left(\max_{1\leq i\leq d}||b_i||_{L^\infty}\right), M^2||c||_{L^\infty}\right\}.$$

We have now shown the boundedness of a. We now show coercivity that is $|a(v, v)| \geq \alpha||v||_V^2$.

$$a(v, v) = \int_\Omega a|\nabla v|^2 + b \cdot \nabla vv + cv^2dx = \int_\Omega a|\nabla v|^2 + \frac{1}{2}b \cdot \nabla(v^2) + cv^2dx$$

Note: $\nabla \cdot (bv^2) = v^2\nabla \cdot b + b \cdot \nabla(v^2)$. Also since v is zero on Γ since $v \in H_0^1$ the divergence theorem gives us that

$$\int_\Omega \nabla \cdot (bv^2)dx = \int_\gamma b \cdot nv^2ds = 0 \Rightarrow \int_\Omega b \cdot \nabla(v^2)dx = -\int_\Omega v^2\nabla \cdot bdx.$$

Thus we have that

$$a(v, v) = \int_\Omega a|\nabla v|^2 - \frac{1}{2}v^2\nabla \cdot b + cv^2dx = \int_\Omega a|\nabla v|^2 + (c - \frac{1}{2}v^2\nabla \cdot b)v^2dx$$

$$\geq \int_\Omega a|\nabla v|^2dx \geq a_0\int_\Omega |\nabla v|^2dx = a_0|v|_1^2,$$

(here we used that $c - \frac{1}{2}\nabla \cdot b \geq 0$) this means a is coercive. Finally, we need to show that L is bounded, that is show $\exists C > 0 : |L(v)| \leq C||v||_V$). We have

$$|L(v)| = |(v, f)| \overset{\text{C.S}}{\leq} ||v||||f|| \overset{\text{Poincaré}}{\leq} C||f|||v|_1 \Rightarrow \frac{|L(v)|}{|v|_1} \leq C||f||$$

$$\Rightarrow ||L||_{V^*} = \sup_{v\in V}\frac{|L(v)|}{|v|_1} \leq C||f||.$$

Which shows that L is bounded. Now by the Lax-Milgram lemma there is a unique $w \in V = H_0^1$ such that $a(w,v) = L(v) \forall v \in V = H_0^1$ and $|w|_1 = ||w||_V \leq C||L||_{V^*} \leq K||f||$. $\square$

When $b = 0$ the bilinear form a is symmetric, then the unique weak solution can be characterized as the minimizer of the energy functional $F(v) = \frac{1}{2}a(v,v) - L(v)$.

Theorem: (Dirichlet's principle) Suppose that $b = 0$, a, c are smooth in $\overline{\Omega}$ and $a(x) > a_0 > 0$ $c(x) >\geq 0$ $x \in \Omega$ then the unique solution of BVP satisfies $F(u) \leq F(v)$ $\forall v \in H_0^1$ where

$$F(v) = \frac{1}{2}\int_\Omega a|\nabla v|^2 cv^2 dx - \int_\Omega fvdx$$

with equality only if $v = u$.

Proof: Theorem A.2 (in the book) shows that $F(u) \leq F(v)$ $\forall v \in V = H_0^1$ as u is a weak solution. If $w \in H_0^1$ such that $F(w) \leq F(v)$ for all $v \in H_0^1$ then by theorem $A.2$, w is a weak solution. By uniqueness $u = w$. $\square$

Inhomogeneous BVP: Classical formulation: $u \in C^2$ such that $\mathcal{L}u = f$ in Ω, $u = g$ on Γ where f and g are given continuous functions.

We would like to consider this problem when $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$. Weak formulation: Find $u \in H^1$ such that $a(u,v) = L(v)$ for all $v \in H_0^1$ $\gamma u = g$ where $\gamma_H^1 \to L^2(\Gamma)$ is the trace operator

$$a(u,v) = \int_\Omega a\nabla u \cdot \nabla v + b \cdot \nabla uv + cuvdx,$$

$$L(v) = \int_\Omega fvdx.$$

Call this problem BVP1.

Theorem: Suppose that there is an $u_0 \in H^1$ such that $\gamma u_0 = g$. If a, b, c are smooth, $a(x) \geq a_0 > 0$, $c(x) - \frac{1}{2}\nabla b(x) \geq 0$ for all $x \in \Omega$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ then there is a unique weak solution of BVP1.

Proof: We look at the problem: find $w \in H_0^1$ such that $a(w,v) = L(v) - a(w,v)$ for all $v \in H_0^1$. As $a : H_0^1 \times H_0^1 \to \mathbb{R}$ is bounded and coercive (like before), L is bounded, and $V \to a(u_0, v)$ is also bounded on H_0^1. We have that

$$|a(u_0,v)| \leq K||u_0||_1||v||_1.$$

By Lax-Milgram there is a unique $w \in H_0^1$ such that $a(w,v) = L(v) - a(u_0,v)$ $\forall v \in H_0^1$. Then $u := w + u_0$ is a weak solution of BVP1.

$$a(u,v) = a(w,v) + a(u_0,v) = L(v) - a(u_0,v) + a(u_0,v) = L(v).$$

Also

$$\gamma u = \gamma w + \gamma u_0 = 0 + g = g,$$

hence u is a weak solution of BVP1. Uniqueness: Suppose that w_1 and w_2 are weak solutions of BVP1. Let $u = w_1 - w_2$,

$$a(u,v) = a(w_1,v) - a(w_2,v) = L(v) - L(v) = 0 = (0,v) \ \forall v \in H_0^1.$$

$$\gamma u = \gamma w_1 - \gamma w_2 = g - g = 0,$$

hence $u \in H_0^1$. Furthermore u solves $a(u,v) = (0,v)$ for all $v \in H_0^1$. Buth this has a unique solution which has to be u, which that satisfies $|u|_1 < ||f|| = c||0|| = 0 \Rightarrow u = 0$ in $H^1 \Rightarrow w_1 = w_2$. $\square$

Neumann problem: We consider the classical formulation: Find $u \in C^2(\overline{\Omega})$ such that $\mathcal{A}u = -\nabla \cdot (a\nabla u) + cu = f$ in Ω, $\frac{\partial}{\partial n} = 0$ on Γ, where $\frac{\partial}{\partial n} = n \cdot \nabla u$ where n is the unit normal of Γ. Let $u \in C^2(\overline{\Omega})$ be a classical solution and $v \in C^1(\overline{\Omega})$ then

$$\int_\Omega vf dx = \int_\Omega \mathcal{A}uvdx = \int_\Omega -\nabla \cdot (a\nabla u) + cudx = -\int_\Gamma a\nabla u \cdot nvds + \int_\Omega a\nabla u \cdot \nabla v + cuvdx =$$

$$\int_\Omega a\nabla u \cdot v + cuvdx \ \forall v \in C^1(\overline{\Omega}).$$

Here we used that the normal derivative is 0. By limit argument using that $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ we set

$$\int_\Omega a\nabla u \cdot \nabla v + cuvdx = \int_\Omega fvdx \ \forall v \in H^1.$$

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Neumann problem continued (Weak formulation): Find $u \in H^1(\Omega)$ such that

$$\int_\Omega a\nabla u \cdot \nabla v + cuvdx = \int_\Omega dx \ \forall v \in H^1(\Omega),$$

if u is a weak solution and $u \in C^1$ then u is a classical solution. Indeed: reversing the steps before we get

$$\int_\Omega fvdx = \int_\Omega -\nabla\cdot(a\nabla u)v + cuvdx + \int_\Gamma a\frac{\partial u}{\partial n}vds \ \forall v \in H^1.$$

Let first $v \in C_0^1 \subset H^1 \Rightarrow$

$$\int_\Omega fvdx = -\int_\Omega -\nabla\cdot(a\nabla u) + cuvdx \ \forall v \in C_0^1 \Rightarrow$$

$$\int_\Omega (\mathcal{L}u - f)vdx = 0 \ \forall v \in C_0^1.$$

Since C_0^1 is dense in L^2, we get $\mathcal{L}u = f$ a.e. If $u \in C^2(\overline{\Omega}), f \in C(\overline{\Omega}) \Rightarrow \mathcal{L}u(x) = f(x)$ in $\Omega \Rightarrow$

$$\int_\Gamma a\frac{\partial u}{\partial n}vds = 0 \ \forall v \in H^1 \Rightarrow \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma.$$

Theorem: Let a, b, c be smooth in $\overline{\Omega}$, $a(x) \geq a_0 > 0 \ \forall x \in \Omega$, $c(x) \geq c_0 > 0$ and $f \in L^2$. Then the Neumann boundary value problem has a unique weak solution.

Proof: Let

$$a(w,v) = \int_\Omega a\nabla w\nabla v + cwvdx, \ \ v, w \in H^1$$

and let

$$L(v) = \int_\Omega fvdx \ v \in H^1.$$

To Show: There is a unique $u \in H^1 : a(u,v) = L(v)\forall v \in H^1$. To show: a is bounded and coercive (a is clearly symmetric and bilinear!).

Bounded:

$$|a(w,v)| \leq \left|\int_\Omega a\nabla w\cdot\nabla vdx\right| + \left|\int_\Omega cwvdx\right| \overset{\text{C.S}}{\leq} ||a\nabla w||_{L^2}||\nabla v||_{L^2} + ||cu||_{L^2}||v||_{L^2} \leq$$

$$||a||_{L^\infty}||\nabla w||||\nabla v|| + ||c||_{L^\infty}||w||||v|| \leq$$

$$||a||_{L^\infty}(||w|| + ||\nabla w||)(||c|| + ||\nabla v||) + ||c||_{L^\infty}(||w|| + ||\nabla w||)(||v|| + ||\nabla v||) = k||w||_1||v||_1,$$

and where $k = ||a||_{L^\infty} + ||c||_{L^\infty}$.

Coercive:

By the Riesz representation theorem (or more generally by Lax-Milgram)
$\exists! u \in H^1 : a(u,v) = L(v)\forall v \in H^1$