TMA690 Partiella Differentialekvationer

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Lecture notes, and solutions to a selection of homework problems.
1 Lecture 2017.10.30

**Notation:** A multi index $\alpha$ is a vector in $\mathbb{R}^d$ whose components $\alpha_j$ are non-negative integers. The length $|\alpha|$ of $\alpha$ is defined by

$$|\alpha| = \sum_{j=1}^{d} \alpha_j.$$ 

If $v : \mathbb{R}^d \to \mathbb{R}$ we may use the multi index notation to define partial derivatives of order $|\alpha|$:

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_k^{\alpha_k}}.$$

**Example:** $\alpha = (1, 0, 1)$, $|\alpha| = 2$

$$D^\alpha v = \frac{\partial^2 v}{\partial x_1 \partial x_3}.$$

**Notation:** For $\xi \in \mathbb{R}^d$ we define $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_d^{\alpha_d}$.

**Example:** $\alpha = (1, 0, 1)$, $\xi = (\xi_1, \xi_2, \xi_3) \Rightarrow \xi^\alpha = \xi_1 \cdot \xi_3$.

In this course we will mainly consider linear partial differential equations of the form

$$\alpha u = \alpha(x, D)u = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f, \text{ in } \Omega$$

$\Omega$ is an open connected set.

**Definition:** We say that the direction $\xi \in \mathbb{R}^d$, $\xi \neq 0$, is a characteristic direction for the operator $\alpha(x, D)$ at $x$ if

$$\Lambda(\xi) = \Lambda(\xi, x) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha = 0.$$

Note: in the sum we only take $|\alpha| = m$ (principle part).

**Definition:** A $(d-1)$-dimensional surface is given locally as a function $F : \mathbb{R}^d \to \mathbb{R}$, $F(x_1, \ldots, x_d) = 0$. The normal is given as $\nabla F = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right)$ for $x \in \mathbb{R}^d$ on surface.

**Main Examples:**

**Example:** First order scalar equations:

$$\sum_{j=1}^{d} a_j(x) \frac{\partial u}{\partial x_j} + a_0(x) u = f, \quad \left( \sum_{|\alpha| \leq 0} a_\alpha(x) D^\alpha u = f \right)$$

Characteristic equation:

$$\sum_{j=1}^{d} a_j(x) \xi_j = 0 \quad \left( \sum_{|\alpha| = 1} a_\alpha(x) \xi^\alpha = 0 \right)$$

Then $\xi$ is a characteristic direction if $\xi$ is perpendicular to $(a_1(x), \ldots, a_d(x))$.

**Example:** Let

$$\Delta u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$
Poison’s equation: \(-\Delta u = f\). Characteristic equation \(\Lambda(\xi) = -(\xi_1^2 + ... + \xi_d^2) = 0 \Rightarrow \xi = 0\). This means that there are no characteristic directions.

**Example:** Heat equation \(\frac{\partial u}{\partial t} - \Delta u = f\). We consider in \(\mathbb{R}^{d+1}\) with variables \((x,t) x \in \mathbb{R}^d\) and \(t \in \mathbb{R}\). With variables \((\xi,\tau)\) the characteristic equation \(\Lambda(\xi,\tau) = -(\xi_1^2 + ... + \xi_d^2) + \tau^2 = 0\). For example the vector \((0,0,...,0,1)\) is a characteristic direction and the plane \(\tau = 0\) is a characteristic surface.

**Example:** Wave equation: \(\frac{\partial^2 u}{\partial t^2} - \Delta u = f\). Consider in \(\mathbb{R}^{d+1}\) with points \((x,t)\). Characteristic equation with variables \((\xi,\tau)\). \(\Lambda(\xi,\tau) = -\langle \xi \rangle^2 \cdot \mathbf{A} = 0\), \(\tau = \pm |\xi|\). Characteristic directions \((\xi, \pm |\xi|)\), \(\xi \neq 0\) anything.

Characteristic surface: Given \(x \in \mathbb{R}^d\) and \(t \in \mathbb{R}\) consider the cone \(|x-x, t-t|^2 = 0\). \(\nabla F = (2(x_1 - x_1, ..., 2(x_d - x_d), -2(t - t)) = 2(x-x, t-t)\). This is of the form \((\xi, \pm |\xi|) \Rightarrow\) this cone is a characteristic surface.

Classification of 2:nd order PDE’s:

Consider second order PDE with constant coefficients:

\[
\sum_{j,k=1}^d a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^d b_j \frac{\partial u}{\partial x_j} + cu = f
\]

where \(a_{jk} = a_{kj}, a_{jk}, b_j, c\) constants. Characteristic equation

\[
\Lambda(\xi) = \sum_{j,k} a_{jk} \xi_j \xi_k = \xi \cdot \mathbf{A} \xi = \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{bmatrix}
\]

\(A\) is symmetric, we can use the Spectral Theorem

\[
A = PDP^{-1}, \quad P^{-1} = P^T \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_d \end{bmatrix}
\]

We introduce a change of variables \(P\eta = \xi\)

\[
\Lambda(\xi) = \Lambda(P\eta) = P \eta \cdot A \eta \cdot P^T = P \eta \cdot PDP^{-1} \eta = P \eta \cdot PD \eta = P^T P \eta \cdot D \eta = \eta D \eta = \sum_{j=1}^d \lambda_j \eta_j^2.
\]

**Definition:** A differential equation is elliptic if all \(\lambda_j\) has the same sign. It is hyperbolic if all but one \(\lambda_j\) has the same sign and it parabolic if the remaining \(\lambda_j = 0\).

Let \(V\) be a vector space over \(\mathbb{R}\).
**Definition:** An inner product on \( V \) is a function \( V \times V \to \mathbb{R} \) such that

1. \( (\lambda u + \mu v, w) = \lambda (u, w) + \mu (v, w) \quad u, w \in V, \lambda, \mu \in \mathbb{R} \)
2. \( (u, v) = (v, u) \quad u, v \in V \)
3. \( (v, v) > 0 \) for all \( v \in V, v \neq 0 \)

The pair \((V, (\cdot, \cdot))\) is called an inner product space.

**Homework:** Show that the following is true:

(a) \( (v, v) = 0 \Leftrightarrow v = 0 \)

(b) \( (w, \lambda u + \mu v) = \lambda (w, u) + \mu (w, v) \)

**Homework solution:** This is shown by using our three axioms.

(a): We begin by showing \( \Rightarrow \): Let \( v = \lambda u \) where \( u \neq 0 \). Then it follows from axiom (1) that \( (v, v) = (\lambda u, \lambda u) = \lambda (u, u) \). Then we use axiom (2) \( \lambda (u, u) = \lambda (\lambda u, u) = \lambda^2 (u, u) \). Since \( (v, v) = 0 \) it follows that \( \lambda^2 (u, u) = 0 \) but we defined that \( u \neq 0 \) thus it follows from axiom (3) that \( (u, u) > 0 \) which means that \( \lambda^2 = 0 \), which implies that \( v = 0 \).

Now we show \( \Leftarrow \): As before let \( v = \lambda u \) where \( u \neq 0 \). By the same reasoning as before we have that \( (v, v) = \lambda^2 (u, u) \), and that \( (u, u) > 0 \). But since \( v = 0 \) and \( u \neq 0 \), \( \lambda \) has to be 0, which in turn means that \( (v, v) = 0 \).

(b): Axiom (2) gives us that \( (w, \lambda u + \mu v) = (\lambda u + \mu v, w) \), axiom (1) then gives us that \( (\lambda u + \mu v, w) = \lambda (u, w) + \mu (v, w) \). Finally we use axiom (2) again and we recieve \( \lambda (u, w) + \mu (v, w) = \lambda (v, u) + \mu (v, v) \). \( \square \)

**Example:** Let \( C[a, b] \) denote the set of real-valued continous functions on \([a, b]\) with addition \((f + g)(x) = f(x) + g(x)\) and scalar multiplication \((\lambda f)(x) = \lambda f(x)\). Define \((f, g) = \int_a^b f(x)g(x)dx\).

**Homework:** Show that \((C[a, b], (\cdot, \cdot))\) is an inner product space.

**Homework solution:** We have to show that the three axioms hold for all the elements in \( C[a, b] \) with the given inner product.

1: Consider \((\lambda f + \mu g, h)\), where \( f, g, h \) are arbitrary elements in \( C[a, b] \) and \( \lambda, \mu \) are arbitrary real constants. Our inner product gives us \( \int_a^b (\lambda f(x) + \mu g(x))h(x)dx \), we use the linearity of the integral \( \int_a^b (\lambda f(x) + \mu g(x))h(x)dx = \lambda \int_a^b f(x)h(x)dx + \mu \int_a^b g(x)h(x)dx \). Thus axiom (1) holds.

2: Consider \((f, g)\) defined as before. According to our inner product \((f, g) = \int_a^b f(x)g(x)dx = \int_a^b f(x)g(x)dx = \int_a^b (g, f) \). This means that axiom (2) holds.

3: Consider \( f \in C[a, b] \) such that \( f \) isn’t the zero function on our interval. We have that \((f, f) = \int_a^b f(x)^2dx \). \( f(x)^2 \geq 0 \) for all \( x \) and since it isn’t the zero function \( f(x) \) has to have non-zero somewhere, thus \( f(x)^2 > 0 \) somewhere. Since we consider \( f \in C[a, b] \) \( f(x)^2 \) has to be non-zero on atleast some interval in \([a, b]\) and 0 at least zero everywhere else, thus by the definition of the integral \( \int_a^b f(x)^2dx > 0 \Rightarrow (f, f) > 0 \). Axiom (3) holds. \( \square \)
2 Lecture 2017.10.31

Definition: A linear functional is a function $f : V \rightarrow \mathbb{R}$ that is linear $f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$, $\lambda, \mu \in \mathbb{R}$, $u, v \in V$.

Definition: A bilinear form $a : V \times V \rightarrow \mathbb{R}$ is a function such that $a(\lambda u + \mu v, w) = \lambda a(u, w) + \mu a(v, w)$ and $a(w, \lambda u + \mu v) = \lambda a(w, u) + \mu a(w, v)$, $u, v, w \in V$ $\lambda, \mu \in \mathbb{R}$. It is symmetric if $a(u, v) = a(v, u)$ and it is positive definite if $a(v, v) > 0$ for all $v \in V$ such that $v \neq 0$.

Homework: Let $V = (C[a, b], (\cdot, \cdot))$ be an inner product space with the inner product $(f, g) = \int_a^b f(x)g(x)dx$. Show the following:

(a): $F(v) = \int_a^b v(x)dx$ is a linear functional.

(b): $F(v) = v(a)$ is a linear functional.

(c): $a(f, g) = \int_a^b f(x)g(x)(1 + x^2)dx$ is a positive definite bilinear form.

Homework solution: We use the definitions:

(a): Let $v, u$ be elements from $C[a, b]$ and $\lambda, \mu$ elements from $\mathbb{R}$. Now consider $F(\lambda v + \mu u) = \int_a^b \lambda v(x) + \mu u(x)dx = \lambda \int_a^b v(x)dx + \mu \int_a^b u(x)dx$. The integrals evaluate to real numbers. This mapping fulfills the condition defined above, it is linear in its argument and it maps functions to real numbers.

(b): Let $u, v$ and $\lambda, \mu$ be defined as above. Now consider $F(\lambda v + \mu u) = \lambda v(a) + \mu u(a)$. This mapping fulfills the condition defined above, it is linear in its argument and it maps functions to real numbers.

(c): Let $f, g, h \in C[a, b]$ and let $\lambda, \mu \in \mathbb{R}$. We begin by showing it’s a bilinear form.

$a(\lambda f + \mu g, h) = \int_a^b (\lambda f(x) + \mu g(x))h(x)(1 + x^2)dx = \lambda \int_a^b f(x)h(x)(1 + x^2)dx + \mu \int_a^b g(x)h(x)(1 + x^2)dx = \lambda a(f, h) + \mu a(g, h)$. We can see that if it is linear in its first argument $a$ has to be linear in its second argument, following from elementary properties of the integral. To show that it is positive definite we consider $a(f, f) = \int_a^b f(x)^2(1 + x^2)dx$ and let $f$ not be the zero function. With $f \in C[a, b]$ we have that it has to be non-zero on atleast some interval in $[a, b]$, thus $f(x)^2$ is greater than zero on atleast some interval in $[a, b]$ and at least zero everywhere else. Also, $(1 + x^2) > 0$ on $[a, b]$. Thus the integral has to be $> 0$, which means that $a$ is positive definite. \[ Q.E.D. \]

Definition: We say that $u \in V$ and $v \in V$ are orthogonal if $(u, v) = 0$. Notation: $u \perp v$.

Definition: Let $V$ be a vector space over $\mathbb{R}$ then a function $\| \cdot \| : V \rightarrow \mathbb{R}_+$ is a norm on $V$ if:

\begin{align*}
(a) \quad & \|v\| > 0 \quad \forall v \neq 0 \\
(b) \quad & \|\lambda v\| = |\lambda|\|v\| \quad \forall v \in V, \lambda \in \mathbb{R} \\
(c) \quad & \|u + v\| \leq \|u\| + \|v\| \quad u, v \in V
\end{align*}

Note: $v = 0 \iff \|v\| = 0$. The pair $(V, \| \cdot \|)$ is called a normed space.

Homework: Let $V = C[a, b]$ be a vector space with the norm $\|f\| = \sup_{x \in [a, b]} |f(x)| = \max_{x \in [a, b]} |f(x)|$. Show that this is a normed space.

Homework solution: We have to show that the given norm fulfills the axioms given any element from $V$.

(a): $|f| \geq 0$, and since according to the axiom $f$ can’t be the zero function it has to be $> 0$
atleast on some interval. If we take the maximum value on that interval we will recieve a real number > 0.

(b): This follows directly from the properties of the supremum/maximum.

\[ \sup_{x \in [a, b]} |\lambda f| = \lambda \sup_{x \in [a, b]} |f|. \]

\[ \sup_{x \in [a, b]} |f + g| \leq \sup_{x \in [a, b]} (|f| + |g|) \leq \sup_{x \in [a, b]} |f| + \sup_{x \in [a, b]} |g|. \]

Thus \(|f + g| \leq |f| + |g|\).

\(\Box\)

If \((V, \langle \cdot , \cdot \rangle)\) is an inner product space then \(|v| = (v, v)^{1/2}\) is a norm.

**Proposition:** Cauchy-Schwartz inequality: Let \((V, \langle \cdot , \cdot \rangle)\) be an inner product space. Then \(|\langle u, v \rangle| \leq ||u|| ||v||\), \(u, v \in V\) with equality if and only if \(u = \lambda v\) for some \(\lambda \in \mathbb{R}\).

**Proof:** If \(v = 0\) the result holds trivially. Let \(t \in \mathbb{R}\) and consider

\[ 0 \leq (u + tv, u + tv) = ||u||^2 + 2\langle u, v \rangle + t^2||v||^2 := f(t). \]

This is a quadratic function, since it’s greater than 0 for all \(t\) it also has to be greater than 0 in its minimum. It can easily be shown that the minimum is \(a = -\frac{\langle u, v \rangle}{||v||^2}\).

\[ 0 \leq f(a) = ||u||^2 - 2\frac{\langle u, v \rangle^2}{||v||^2} + \frac{(u, v)^2}{||v||^2} = ||u||^2 - \frac{(u, v)^2}{||v||^2} \Rightarrow (u, v)^2 \leq ||u||^2 ||v||^2 \Rightarrow |(u, v)| \leq ||u|| ||v|| \]

If \(u = -tv\) we have equality. \(\Box\)

**Proposition** Triangle inequality: \(|u + v| \leq |u| + |v|\).

**Proof:** We prove this by using Cauchy-Schwartz inequality

\[ ||u + v||^2 = (u + v, u + v) = ||u||^2 + 2\langle u, v \rangle + ||v||^2 \leq ||u||^2 + 2||u|| ||v|| + ||v||^2 \]

\[ (||u|| + ||v||)^2 \Rightarrow ||u + v|| \leq ||u|| + ||v|| \]

\(\Box\)

**Homework:** Prove the Parallellogram identity: \(|u + v|^2 + |u - v|^2 = 2(||u||^2 + ||v||^2)\)

**Homework solution:** We simply use the axioms and the definition of the norm!

\[ ||u + v||^2 + ||u - v||^2 = (u + v, u + v) + (u - v, u - v) = (u, u + v) + (v, u + v) + (u, u - v) - (v, u - v) = (u, u) + (u, v) + (v, u) + (v, v) + (u, u) - (v, u) - (v, v) = 2(||u||^2 + ||v||^2) \]

\(\Box\)

**Definition:** Let \((x_n) \subset V\) be a sequence in \((V, ||\cdot||)\), we say \(x_n \to x \in V\) as \(n \to \infty\) alternatively written as \(\lim_{n \to \infty} x_n = x\) if \(\lim_{n \to \infty} ||x_n - x|| = 0\), with \(\varepsilon - \delta\)-notaion:

\(\langle \forall \varepsilon > 0 \rangle (\exists N) : n \geq N \Rightarrow ||x_n - x|| < \varepsilon.\)

**Definition:** A sequence is a Cauchy-sequence if \((\forall \varepsilon > 0)(\exists N) : m, n \geq N \Rightarrow ||x_n - x_m|| < \varepsilon\). It can be stated informally as: \(\lim_{m,n \to \infty} ||x_n - x_m|| = 0\).

**Fact:** If \((x_n)\) is convergent then \(x_n\) is a Cauchy-sequence. \(\mathbb{Q}\) The converse is not true! \(\mathbb{Q}\)

A normed space is called complete if every Cauchy-sequence converges. A complete normed space is called a Banach space and a complete inner product space is called a Hilbert space.

**Example:** \(C[a, b], ||f|| = \sup_{x \in [a, b]} |f|\) is a Banach space.
Homework: Show that $C[a,b], ||f|| = \int_a^b f(x)^2 \frac{1}{2}$ is not complete.

Homework solution:
Find a function that is Cauchy but that doesn’t converge to a continous function. Try a function which converges to a step function.

Example:

\[ V = \{(x_n)\}, \quad x_n \in \mathbb{R}, \quad \sum_{n=1}^{\infty} |x_n|^2 < \infty, \quad ((x_n),(y_n)) = \sum_{n=1}^{\infty} x_n \cdot y_n \]

\((V,(\cdot,\cdot))\) is complete.

Definition: Let \(V,W\) be normed spaces. A mapping \(B : V \to W\) is linear if \(B(\lambda u + \mu v) = \lambda Bu + \mu Bv\) \(u,v \in V, \lambda,\mu \in \mathbb{R}\). It is bounded if there is \(c > 0\) such that \(||Bv||_W \leq c||v||_V\) for all \(v \in V\). We may then define the norm of \(B\) by

\[ ||B|| = \sup_{v \in V, v \neq 0} \frac{||Bv||_W}{||v||_V} = \sup_{||v||_V = 1} ||Bv||_W = \inf\{c \in \mathbb{R} : ||Bv||_W \leq c||v||_V\text{ for all } v \in V\} \]

\[ \Rightarrow ||Bv||_W \leq ||B|| \cdot ||v||_V \]

Homework: Show the equalities above.

Homework solution:

Definition: We denote the set of bounded linear operators by \(\mathcal{B}(V,W)\) if \(V = W, \mathcal{B}(V)\). This can be made to be a vector space:

\[ (B_1 + B_2)v = B_1v + B_2v \quad v \in V \]

\[ (\lambda B)v = \lambda Bv \quad \lambda \in \mathbb{R}, \quad v \in V \]

Then \(\mathcal{B}(V,W)\) is a normed space and if \(W\) is complete so is \(\mathcal{B}(V,W)\).

Homework: Show that \(||B||\) defined as above is a norm.

Homework solution:

Lemma: \(B \in \mathcal{B}(V,W)\) \(\iff\) \(B\) is continuous that is \(x_n \to x \Rightarrow Bx_n \to Bx\).

Definition: Let \(V\) be a normed space. The space of continuous linear functionals is \(\mathcal{B}(V,\mathbb{R})\). Notation: \(V^* = \mathcal{B}(V,\mathbb{R})\), \(V^*\) is called the dual space of \(V\). Since \(\mathbb{R}\) is complete so is \(V^*\).

A bilinear form \(a : V \times V \to \mathbb{R}\) is bounded if there is \(c > 0\) sicj that \(|a(u,v)| \leq c||u|| \cdot ||v||\).

Definition: The ball centered at \(v_0 \in V\) with radius \(r > 0\) is \(B_r(v_0) = \{v \in V : ||v - v_0|| < r\}\).

Definition: A set \(A \subset V\) is open if for every \(v_0 \in A\) there is \(r = r(v_0)\) such that \(B_r(v_0) \subset A\).

Definition: \(A\) is closed if \(A^c = V \setminus A\) is open.

Homework: Show that \(A\) is closed \(\iff (x_n) \in A, x_n \to x \in V \Rightarrow x \in A\).

Homework solution:

Definition: \(A \subset V\) is a dense subset of \(V\) of for all \(v \in V\) there is \(v_n \in A \quad v_n \to v\).

Theorem: Let \(V\) be a Hilbert space and \(V_0 \subset V\) be a closed subspace. Then any \(v \in V\) can be uniquely be written as \(v = v_0 + w\) where \(v_0 \in V_0\) and \(w \perp v_0\). The element \(v_0\) can be

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characterised as the unique element in $V_0$ such that $||v - v_0|| = \min\{||v - u||, u \in V_0\}$. The element $v_0$ is denoted by $P_{V_0}v$. 

\[ V_0 \]
3 Lecture 2017.11.06

Corollary: $V$ is a Hilbert space, $V_0 \subset V$ is a closed subspace, $V_0 \neq V$. Then $w \in V \setminus V_0$, $w \perp v_0$

Proposition: $V_0 \neq V \Rightarrow \exists v_0 \in V \setminus V_0$, $w_0 \neq 0$. Projection theorem:

\[ w_0 = v_0 + w, \quad w = v_0 \perp w \neq 0 \text{ as } w_0 \neq v_0. \]

Theorem: (Riesz Representation Theorem) Let $V$ be a Hilbert space and $L : V \to \mathbb{R}$ be a bounded linear functional on $V$ (i.e. $L \in V^*$). Then there is a unique $u \in V$ such that $L(V) = (v, u)$ for all $v \in V$. Furthermore $\|L\|_V = \|u\|$.

Proof: See the book.

Note: The Riesz representation theorem identifies continuous linear functionals with elements of the Hilbert space $V$.

Homework: Show that the map $\Phi : L \to u (V^* \to V)$ is linear, surjective and isometric. $(V$ and $V^*$ are isometrically isomorphic).

Homework solution:

Often in this course we will study the following problem: Let $V$ be a Hilbert space and $L : V \to \mathbb{R}$ be a bounded and $a : V \times V \to \mathbb{R}$ bilinear positive definite. Problem: Find $u \in V$ such that $a(u, v) = L(v)$ for all $v \in V$. Call this problem $(V)$.

Definition: A bilinear form $a : V \times V \to R$ is called coercive of there is an $\alpha > 0$ such that $a(v, v) \geq \alpha \|v\|^2$ for all $v \in V$. Note that coercive implies positive definite, but positive definite does not imply coercive. In finite dimensions however, positive definite and coercive is equivalent.

If $a : V \times V \to \mathbb{R}$ is positive definite, symmetric and bilinear, then $a$ is an inner product on $V$.

If $a$ is coercive and bounded, then the norm (energy norm) $\|v\|_a = a(v, v)^{1/2}$ is equivalent to the original norm $\|\cdot\|$. $\alpha \|v\|^2 \leq a(v, v) \leq M \|v\|^2$.

In summary: If $a : V \times V \to \mathbb{R}$ is bilinear, coercive, symmetric and bounded then: the energy norm $\|\cdot\|_a$ and $\|\cdot\|$ are equivalent and therefore $(V, \|\cdot\|_a)$ is complete (hence a Hilbert space). Also $L$ is bounded linear on $(V, \|\cdot\| \Rightarrow)$ bounded linear on $(V, \|\cdot\|_a)$.

In this case the Riesz representation theorem on $(V, \|\cdot\|_a)$ yields that there is an unique $u \in V : L(v) = a(v, u) = a(u, v)$ for all $v \in V$. Thus equation $(V)$ has a unique solution.

Energy estimate: We may bound the norm of the solution in terms of $L$:

\[ \alpha \|u\|^2 \leq a(u, u) = L(u) \leq \|L\|_V \|u\| \Rightarrow \|u\| \leq \|L\|_V^{1/2}. \]

The solution to $(V)$ may be characterized through a minimization problem:

Theorem: If $a : V \times V \to \mathbb{R}$ is symmetric and positive definite then $u$ is a solution to problem $(V) \iff F(u) \leq F(v)$ for all $v \in V$ $F(u) = \frac{1}{2}a(u, u) - L(u)$

Proof: Suppose that $u$ is a solution to $(V)$. Set $w = v - u \Rightarrow v = u + w$. Then

\[ F(v) = F(u + w) = \frac{1}{2}a(u + w, u + w) - L(u + w) = \frac{1}{2}a(u, u) - L(u) + a(u, w) - L(w) + \frac{1}{2}a(w, w) \]

The sum of the first two terms are equal to $F(u)$ by definition. The sum of the second two terms are equal to 0 since $u$ is a solution. Thus we have $F(v) \geq F(u)$ since $a(w, w) \geq 0$.

Now suppose $F(u) \leq F(v)$ for all $v \in V$. Consider $g(t) = F(u + tv) \geq F(u) = g(0)$, where $t$ is a
real parameter we have

\[ g(t) = F(u + tv) = \frac{1}{2} a(u + tv, u + tv) - L(u + tv) = \frac{1}{2} t^2 a(v, v) + (a(u, v) - L(v)) t + \frac{1}{2} a(u, u) - L(u) \]

This is a quadratic in \( t \) and it has a minimum at 0 thus

\[ 0 = g'(0) = a(u, v) - L(v) \Rightarrow a(u, v) = L(v) \]

**Note:** \( F \) is called the energy functional and \( (V) \) the variational equation for \( F \).

There is an extension when \( a \) is non-symmetric.

**Theorem:** (Lax-Milgram) Let \( V \) be a Hilbert space and \( a : V \times V \to \mathbb{R} \) be a bounded coercive bilinear form and \( L : V \to \mathbb{R} \) be a bounded linear functional then there is a unique \( u \in V \) such that

\[ a(u, v) = L(v) \]

for all \( v \in V \). (That is \( (V) \) has a unique solution)

**Note:** Unlike the symmetric case before there is no characterization of \( u \) through the minimization of an energy functional. But we still have

\[ ||u|| \leq \frac{1}{\alpha} ||L||_{V^*}. \]

**Function spaces:** Let \( \Omega \subset \mathbb{R}^d \) then \( \overline{\Omega} \) denotes the closure of \( \Omega \).

\[ \overline{\Omega} = \bigcap_{\Omega \subset A, A \text{ is closed}} A \]

An example is that the closure of a ball is the ball with its boundary.

Let \( \Omega \) be a domain \( \equiv \) open, connected. \( C(\Omega) : \text{vector space of continuous functions } \Omega \to \mathbb{R} \).

If \( \Omega \) is bounded domain then \( C(\overline{\Omega}) \) is a Banach space with norm

\[ ||V||_{C(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} |v(x)| = \max_{x \in \overline{\Omega}} |v(x)| \]

\( C^k(\Omega) : \text{space of } k\text{-times continuously differentiable functions on } \Omega : \text{then } D^\alpha v \text{ is continuous for all } |\alpha| \leq k. \)

\( C^k(\Omega) : \{ v \in C^k(\Omega) : D^\alpha v \in C(\overline{\Omega}), |\alpha| \leq k \}. \) This is a Banach space if we set

\[ ||v||_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq k} ||D^\alpha v||_{C(\overline{\Omega})}. \]

In 1D: \( \Omega = (0, 1) \):

\[ ||v||_{C^k(\Omega)} = \sup_{x \in [0,1]} |v(x)| + \sup_{x \in [0,1]} |v'(x)| + \sup_{x \in [0,1]} |v''(x)| \]

A function \( V : \Omega \to \mathbb{R} \) has compact support if \( v = 0 \) outside of a compact set (compact \( \iff \) bounded and closed in \( \mathbb{R}^d \))

\( C^k_0(\Omega) \) is the space of functions in \( C^k(\Omega) \) with compact support.

\( C^\infty_0(\Omega) : v \in C^k_0(\Omega) \) for every \( k. \)
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**Definition:** Let $\Omega \subset \mathbb{R}^d$ be a domain. To begin with let, $1 \leq p < \infty$. A function $v \in L^p(\Omega)$ if $\int_{\Omega} |v(x)|^p dx < \infty$. We define $\|v\|_{L^p(\Omega)} = \left(\int_{\Omega} |v(x)|^p dx\right)^{1/p}$. Here follows a couple of notes regarding this definition.

**Note 1:** Here $\int_{\Omega} f(x) dx$ denotes the Lebesgue integral. It coincides with the Riemann integral for bounded Riemann integrable functions (at least on bounded $\Omega$). For such functions the Lebesgue integral is an extension of the Riemann integral.

**Note 2:** There are many functions that are not Riemann integrable but are Lebesgue integrable.

**Example:** $\Omega = (0,1)$, consider the Dirichlet-function:

$$v(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Note that $v$ is very simple $v = \chi_{\mathbb{Q}\cap(0,1)}$. It’s easy to see that $v$ is not Riemann integrable, however it is Lebesgue integrable and $\int_{\Omega} v(x) dx = 0$.

**Note 3:** The Lebesgue integral behaves much nicer than the Riemann integral if one wants to exchange limits and integrals.

**Example:** Suppose $f_n(x) \to f(x), f \in \Omega$. Then $|f_n(x)| \leq g(x), g(x) \in L^1(\Omega) \Rightarrow \int_{\Omega} f(x) dx = \lim_{n \to \infty} \int_{\Omega} f_n(x) dx$. This is called Lebesgue’s dominated convergence theorem.

**Note 4:** We consider two functions $v$ and $w$ equivalent, or we say that they are equal almost everywhere (a.e) if $v(x) = w(x)$ only for $x \in A$ where $A$ has Lebesgue measure 0, defined as follows: Let $c = (a_1, b_1) \times \ldots \times (a_d, b_d) \subset \mathbb{R}^d$ be a hypercube in $\mathbb{R}^d$. The Lebesgue measure $m(c)$ of $c$ is defined by $m(c) = \prod_{i=1}^d (b_i - a_i)$.

**Definition:** A set $A \subset \mathbb{R}^d$ has Lebesgue measure 0 if for every $\epsilon > 0$ there are countably many hypercubes $c_n$, $n = 1, 2, \ldots$ such that $A \subset \bigcup_{n=1}^\infty c_n$ and $\sum_{n=1}^\infty m(c_n) < \epsilon$. Note that if $A = \{a\}$ then $m(A) = 0$, if $A$ is countable then $m(A) = 0$.

**Example:** Consider $\mathbb{R}^2$ then the real line $A = \{(x,0), x \in \mathbb{R}\}$ has Lebesgue measure 0 (a line has 0 "area"). In general if $\Omega \subset \mathbb{R}^d$ a domain, then the boundary $\Gamma$ of $\Omega$ ($\Gamma = \partial \Omega \cap \Omega$) has Lebesgue measure 0. For example $\{(x,0), x \in \mathbb{R}\} = \Gamma$, $\Omega = \{(x,y): x \in \mathbb{R} y > 0\}$.

**Note 5:** If $v = w$ a.e, then if $v$ is Lebesgue integrable then so is $w$ and $\int_{\Omega} v dx = \int_{\Omega} w dx$.

**Example:** With the Dirichlet function from before $v \equiv 0$ a.e because $m(\mathbb{Q} \cap (0,1)) = 0$ thus $v$ is Lebesgue integrable with Lebesgue integral 0.

**Note 6:** Elements of the space $L^p(\Omega)$ are equivalence classes of functions that are equal a.e. Therefore in general we cannot talk about point values of $v \in L^p(\Omega)$, that is $v(x)$ for fixed $x$ (unless there is a continous representation in the equivalence class).

**Note 7:** $L^p(\Omega)$ is complete and hence a Banach space. $p = 2$, $L^2(\Omega)$ is a Hilbert space with inner product $(u,v) = \int_{\Omega} u v dx$ where this is the Lebesgue integral.

**Note 8:** Regarding $p = \infty$. We say that $v$ is essentially bounded if there is a $M > 0$ such that $|v(x)| \leq M$ for almost all $x$.

$$\|v\|_{L^\infty} = \inf \{M : |v(x)| \leq M \text{ for almost all } x\} = \sup_{x \in \Omega} |v(x)|$$

$L^\infty$ is a Banach space.
Example: $\Omega = (0, 1)$ and for $n = 1, 2, ...$

\[
\begin{cases}
1 & \text{if } x \neq \frac{1}{n} \\
0 & \text{if } x = \frac{1}{n}
\end{cases}
\]

Suppose $\sup_{x \in \Omega} |v(x)| = \infty$ but ess sup$_{x \in \Omega} |v(x)| = 1$.

Note 9: If the boundary $\Gamma$ of $\Omega$ is smooth enough (say, Lipschitz continuous) then $C^0_0(\Omega)$ (also $C^\infty_0(\Omega)$) is dense in $L^p(\Omega)$, $1 \leq p < \infty$. That is for every $v \in L^p(\Omega)$ there are $(v_n) \subset C^0_0(\Omega)$ or $(v_n) \subset C^\infty_0(\Omega)$ such that $||v_n - v||_{L^p} \to 0$ as $n \to \infty$. This does not hold for $L^\infty$.

Sobolev spaces: We need the concept of weak (or generalized or distributional) derivatives. We begin with a lemma.

Lemma: Suppose that $V$ and $W$ are Banach spaces and $A \subset V$ is a dense subspace of $V$ (dense: $\forall v \in V \exists (v_n) \subset A: v_n \to v$). Suppose that $B: A \to W$ is a bounded linear operator. Then there is a unique linear continuous (≡ bounded) extension $\tilde{B}$ of $B$ to the whole of $V$ such that $||\tilde{B}||_{B(V,W)} = ||B||_{B(A,W)}$.

Let $\Omega \subset \mathbb{R}^d$ be a domain. Let $v \in C^1(\overline{\Omega})$. Let $\Phi \in C^1_0(\Omega)$. Integrate by parts:

\[
(*) = \int_{\Omega} \frac{\partial v}{\partial x_i} \Phi \, dx = - \int_{\Omega} v \frac{\partial \Phi}{\partial x_i} \, dx
\]

This is a special case of Greens formula (see introduction of the book) $w = (w_1, ..., w_d)$ vector field, $\psi$ scalar field then

\[
\int_{\Omega} w \cdot \nabla \psi \, dx = \int_{\Gamma} w \cdot n \psi \, dx - \int_{\Omega} \nabla w \psi \, dx
\]

$n$ is the outward facing unit normal of $\Gamma$.

If $v \in L^2(\Omega)$ it might not have a classical derivative. One can define the generalized (weak) derivative denoted by $\frac{\partial v}{\partial x_i}$ to be a functional with the following properties:

Definition: The weak derivative is defined as

\[
\frac{\partial v}{\partial x_i}(\Phi) = L(\Phi) = - \int_{\Omega} v \frac{\partial \Phi}{\partial x_i} \, dx, \; \Phi \in C^1_0(\Omega)
\]

Suppose that $L$ is bounded that is there is a $M > 0$ such that $|L(\Phi)| \leq M ||\Phi||_{L^2} \forall \Phi \in C^1_0(\Omega)$. Then by the lemma there is a continuous linear extension of $L$ to the whole of $L^2$ (because $C^1_0$ is dense in $L^2$). By Riesz representation theorem there is an unique $w \in L^2$ such that $L(\Phi) = (\Phi, w)$ $\Phi \in L^2$. Therefore in this case

\[
\int_{\Omega} v \frac{\partial \Phi}{\partial x_i} \, dx = L(\Phi) = \int_{\Omega} \Phi w \, dx \forall \Phi \in C^1_0(\Omega)
\]

In this case we say that $\frac{\partial v}{\partial x_i}$ is in $L^2$. We still denote $w$ by $\frac{\partial v}{\partial x_i}$. With this notation

\[
(**) = - \int_{\Omega} v \frac{\partial \Phi}{\partial x_i} \, dx = \int_{\Omega} \Phi \frac{\partial v}{\partial x_i} \, dx \forall \Phi \in C^1_0(\Omega)
\]

Comparing $(*)$ with $(**)$ we say that for $v \in C^1_0(\overline{\Omega})$ the weak derivative coincides with the classical derivative. Note: weak derivative allows for integration by parts in the appropriate way.
Let $\alpha$ be a multiindex and $v \in L^2(\Omega)$. Define $D^\alpha v$ as a functional:

$$(D^\alpha v)(\Phi) = L(\Phi) = (-1)^{|\alpha|} \int_\Omega v D^\alpha \Phi dx, \quad \Phi \in C_0^{[\alpha]}(\Omega)$$

If $|L(\Phi)| \leq ||\Phi||_{L^2}$ then since $\Phi \in C_0^{[\alpha]}(\Omega)$ is dense, there is a unique continuous extension of $L$ to the whole of $L^2$. By the Riesz representation theorem there is $w \in L^2$ which we denote by $D^\alpha v$ such that $(w, \phi) = (D^\alpha v, \Phi) = L(\Phi) = (-1)^{|\alpha|} \int_\Omega v D^\alpha \Phi dx = (-1)^{|\alpha|}(v, D^\alpha \Phi)$, $\forall \Phi \in C_0^{[\alpha]}(\Omega)$.

**Definition:** The Sobolev space $H^k(\Omega)$ is defined by:

$$H^k(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega) \mid |\alpha| \leq k\}$$

We endow $H^k$ with the inner product

$$(u, v)_{H^k} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v dx$$

and with the norm:

$$||u||_{H^k} = ||u||_k = \left( \sum_{|\alpha| \leq k} \int_\Omega (D^\alpha u)^2 dx \right)^{1/2}$$

**Note:** For $H^0$ we have $||v||_0 = ||v||_{H^0} = ||v||_{L^2} = ||v||$. For $H^1$ we have:

$$||v||_1 = \left( \int_\Omega v^2 + \sum_{j=1}^d \left( \frac{\partial v}{\partial x_j} \right)^2 dx \right)^{1/2}$$

and for $H^2$ we have:

$$||v||_2 = \left( \int_\Omega v^2 + \sum_{j=1}^d \left( \frac{\partial v}{\partial x_j} \right)^2 + \sum_{j=1}^d \sum_{k=1}^d \left( \frac{\partial^2 v}{\partial x_j \partial x_k} \right) dx \right)^{1/2}$$

note that the $H^2$ norm contains all the mixed second order derivatives not just the Laplacian!

We continue by listing two important properties of the Sobolev spaces.

**Property 1:** $H^k$ is a Hilbert space

**Property 2:** $C^l(\overline{\Omega})$ is a dense subspace of $H^k(\Omega)$ for $l \geq k$, this holds if $\Gamma = \partial \Omega$ is smooth enough.

**Definition:** The seminorm $|\cdot|_k$ is defined by:

$$|v|_k = \left( \sum_{|\alpha| = k} \int_\Omega (D^\alpha v)^2 dx \right)^{1/2}$$

This is not a norm, for example $|v|_k = 0$ for $v$ constant. Still the triangle inequality holds and $|\lambda v|_k = |v|_k$.

**Definition:** We define the trace. This is the generalization of the boundary value of a function. If $v \in C^k(\overline{\Omega})$ then we may define the boundary value $\gamma v$ of $v$ by restricting $v$ to $\Gamma : (\gamma v)(x) = v(x) \ x \in \Gamma$. Then $\gamma v$ is a continuous function on $\Gamma$. We would like to extend this concept to $v \in H^1$. 12.
**Problem:** $\Gamma$ has the Lebesgue measure 0 in $\mathbb{R}^d$. As functions in $H^1$ are only defined as $L^2$ functions the point values on $\Gamma$ are not well defined.

**Idea:** We define the boundary space $L^2(\Gamma)$ as the space of functions on $\Gamma$ such that the surface integral $\int_\Gamma v^2\,ds < \infty$, with the norm $||v||_{L^2(\Gamma)} = \left(\int_\Gamma v^2\,ds\right)^{1/2}$. We will first define the boundary value of a function $v \in C^1(\Omega) \subset H^1$ by restriction of $v$ to the boundary and we try to extend this notion to the whole of $H^1$ using the denseness of $C^1(\Omega)$ in $H^1$.

**Lemma:** Let $\Omega = (0,1)$. Then there is a constant $c > 0$ such that $|v(x)| \leq C||v||_1$ for all $c \in C^1(\overline{\Omega})$ and $x \in \overline{\Omega}$ (in particular we may take $x = 0, 1$).

**Proof:** For $x, y \in \Omega$ and $v \in C^1(\Omega)$ we have $v(x) = v(y) + \int_y^x v'(s)\,ds$ (this is nothing but usage of the fundamental theorem of integral calculus). Then we use the triangle inequality, the triangle inequality for integrals and Cauchy-Schwarz

$$|v(x)| \leq |v(y)| + \int_y^x |v'(s)|\,ds \leq |v(y)| + \int_y^x 1 \cdot |v'(s)|\,ds \leq |v(y)| + \left(\int_0^1 1^2\,ds\right)^{1/2} \left(\int_0^1 |v'(s)|^2\,ds\right)^{1/2}$$

The limits of integration can change from $x, y$ to 0, 1 since the absolute value makes the integral grow when the interval grows, thus it is fine to make enlarge our limits to the whole of $\Omega$ in our inequality. Then we use $(a + b)^2 \leq 2a^2 + 2b^2$:

$$|v(x)|^2 \leq 2\left(|v(y)|^2 + \int_0^1 |v'(s)|^2\,ds\right)$$

Since the righthand side is independent of $y$ and the second term on the lefthand side is independent of $y$ we can take the integral with respect to $y$ on both sides (since the length of our integral is 1 these objects integrate like multiplication with 1) and acquire

$$|v(x)|^2 \leq 2\left(||v||_{L^2}^2 + ||v'||_{L^2}^2\right) = 2||v||_1^2$$

By continuity this result holds for $x \in \overline{\Omega}$. We have $|v(1)| = \lim_{x_n \to x} |v(x)|$ and $x_n \to x$, $|x_n| \leq m \Rightarrow |x| \leq m$. This concludes the proof. 

**Theorem:** (Trace theorem) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Suppose that $\Gamma = \partial\Omega$ is a polygon or smooth. We define the trace operator $\gamma$ by $\gamma : C^1(\overline{\Omega}) \subset H^1(\Omega) \to C^1(\Gamma) \subset L^2(\Gamma)$ $(\partial v)(x) = v(x)$ $x \in \Gamma$. Then there is a bounded linear extension of $\gamma$ to the whole of $H^1(\Omega)$ still denoted by $\gamma$. In particular there is a $c > 0$ such that $||\gamma v||_{L^2(\Gamma)} \leq c||v||_{H^1(\Omega)} \forall v \in H^1(\Omega)$.

**Note:** In this setting the "boundary value" of a function in $H^1(\Omega)$ only exists as a function on $L^2(\Gamma)$.

**Proof:** $\gamma$ is clearly linear. By homework problem 2.5 we only need to show that $||\gamma v||_{L^2(\Gamma)} \leq c||v||_{H^1}$ $v \in C^1(\overline{\Omega})$ as $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$. We will prove this for $(0,1) \times (0,1)$ We will only consider one side of the rectangle, the same reasoning as follows holds for the other three. Let $(x_1, x_2) \in \Omega$ we use the lemma applied to the function $x \to v(x_1, x_2)$ and $x_1 = 0$ (right side of the rectangle).

$$v(0, x_2)^2 \leq 2\left(\int_0^1 v(x_1, x_2)^2\,dx_1 + \int_0^1 \left(\frac{\partial v(x_1, x_2)}{\partial x_1}\right)^2\,dx_1\right)$$

$$\int_0^1 v(0, x_2)^2\,dx_2 \leq 2\left(\int_0^1 \int_0^1 v(x_1, x_2)^2\,dx_1\,dx_2 + \int_0^1 \int_0^1 \left(\frac{\partial v(x_1, x_2)}{\partial x_1}\right)^2\,dx_1\,dx_2\right) \leq 2\left(||v||_{L^2(\Omega)}^2 + ||\nabla v||_{L^2(\Omega)}^2\right)$$

This implies that $||v||_{L^2(\Gamma)} \leq 2||v||_1^2$ 

**Definition:** We saw that the trace operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is bounded and therefore it’s nullspace (kernel) is a closed subspace of $H^1_0$. We define $H^1_0$:

$$H^1_0(\Omega) = \{v \in H^1(\Omega) | \gamma v = 0\}$$
It is a closed subspace of $H^1$ these are all the functions in $H^1$ that vanish on the boundary $\Gamma$ in the trace sense.

**Homework:** $T : V \rightarrow W$, where $V$ and $W$ are normed spaces, is bounded. Show that $\ker(T) = \{ v \in V : Tv = 0 \}$ is a closed subspace of $V$.

**Homework solution:**
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Theorem: (Poincaré inequality) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Then there is a constant $C$ such that $||v||_{L^2} \leq C||\nabla v||_{L^2}$ for all $v \in H^1_0(\Omega)$. It is important that $v \in H^1_0$ (zero on boundary).

Proof: Fact: $C^1_0$ is dense in $H^1_0$ therefore it is enough to prove that $||v||_{L^2} \leq C||\nabla v||_{L^2}$ for all $v \in C_0^1(\Omega)$. Indeed: $v \in H^1_0$, $v_n \in C_0^1$ and $v_n \to v$ in $H^1$-norm $v_n \to v$ in $L^2$, $\nabla v_n \to \nabla v$ in $L^2$ imply:

$||v_n||_{L^2} \leq C||\nabla v_n||_{L^2} \to ||v||_{L^2} \leq ||\nabla v||_{L^2}$ as $n \to \infty$

as $|| \cdot ||_{L^2}$ is continuous. We will prove this for $\Omega = (0, 1) \times (0, 1)$. Let $v \in C^1_0(\Omega)$ $x \in (x_1, x_2) \in \Omega$. Then:

$v(x_1, x_2) - v(0, x_2) = \int_0^{x_1} \frac{\partial v}{\partial x_1}(s, x_2)ds$

This is simply the fundamental theorem of calculus. The second term on the righthand side is 0 because of compact support. We now use Cauchy-Schwarz, our second factor is the invisible 1 in front of our derivative of $v$:

$v(x_1, x_2)^2 \leq \int_0^{x_1} 1^2 ds \cdot \int_0^{x_1} \left(\frac{\partial v}{\partial x_1}(s, x_2)\right)^2 ds \leq \int_0^1 \left(\frac{\partial v}{\partial x_1}(s, x_2)\right)^2 ds$.

Here the last inequality follows from $x_1 \leq 1$, since we have a squared real valued function the integral can only get bigger if we extend our integration limits. We now integrate the above inequality over all of $\Omega$:

$\int_0^1 \int_0^1 v(x_1, x_2)^2 dx_1 dx_2 \leq \int_0^1 \int_0^1 \left(\frac{\partial v}{\partial x_1}(s, x_2)\right)^2 ds dx_2$.

The integral over $x_1$ on the righthand side evaluates to 1 since the righthand side doesn’t depend on $x_1$. The righthand side definitely is smaller than the norm of the gradient squared, if we add more derivative terms we will end up with something larger. Thus we have:

$\int_0^1 \int_0^1 v(x_1, x_2)^2 dx_1 dx_2 \leq \int_0^1 \int_0^1 \left(\frac{\partial v}{\partial x_1}(s, x_2)\right)^2 ds dx_2 \leq ||\nabla v||_{L^2}^2$,

which we wanted to show. \hfill \Box

Corollary: If $v \in H^1_0$ then:

$|v|^2 = ||\nabla v||_{L^2}^2 \leq ||v||_{L^2}^2 + ||\nabla v||_{L^2}^2$ (as $||v||_{L^2}^2 \leq C||\nabla v||_{L^2}^2$)

Therefore on $H^1_0$ $| \cdot |_{L^2}$ and $|| \cdot ||_{L^2}$ are equivalent and thus $| \cdot |_{L^2}$ is a norm on $H^1_0$ not just a seminorm.

Definition: The dual space $(H^1_0)^*$ is denoted by $H^{-1}$. That is $H^{-1}$ is the space of bounded linear functionals on $H^1_0$. If we equip $H^1_0$ with $| \cdot |_{L^2}$ then the norm on $H^{-1}$ is given by

$||L||_{H^{-1}} = \sup_{v \in H^1_0} \frac{|L(v)|}{|v|_{L^2}}$.

Boundary value problems: We will consider a general second order elliptic problem of the form (which we will refer to as BVP):

$L u = -\nabla \cdot (a \nabla u) + b \cdot \nabla u + cu = f$

where $f \in \Omega \subset \mathbb{R}^d$ and $u = 0$ on $\Gamma$. $a$, $b$ and $c$ are smooth functions (b vectorfield) and $f$ is continuous.

Definition: A function $u$ is a classical solution of the boundary value problem if $u \in C^2(\overline{\Omega})$ and $u$ satisfies BVP.
Note: In applications one would like to consider more general \( f \), say \( f \in L^2 \). We need a more general solution concept, weak or variational formulation of BVP.

Suppose that \( u \in C^2(\Omega) \) is a classical solution. We take \( v \in C^0_0(\Omega) \) multiply both sides of the equation BVP by \( v \) and integrate over \( \Omega \) (note: integration by parts):

\[
\int_\Omega f v = \int_\Omega Lu v = \int_\Omega -\nabla \cdot (a \nabla u) v + b \nabla u v + c uv dx = -\int_\Gamma a \nabla u v n + \int_\Omega a \nabla u \nabla v + b \nabla u v + c uv dx.
\]

The integral over \( \Gamma \) is 0 since \( v \in C^0_0 \). Thus we have we have:

\[
\int_\Omega a \nabla u \cdot \nabla v + b \cdot \nabla u v + c uv dx = \int_\Omega f v dx \quad \forall v \in C^0_0(\Omega)
\]

Claim: This holds for all \( v \in H^1_0(\Omega) \). \( v \in H^1_0 \), \( (v_n) \in C^0_0 \) such that \( v_n \to v \) in \( L^2 \) and \( \nabla v_n \to \nabla v \) in \( L^2 \). Thus our equation can be extended to \( H^1_0 \) by taking the limit \( n \to \infty \), we also note that our integral is a sum of inner products in \( L^2 \):

\[
(a \nabla u, v_n) + (b \cdot \nabla u, v_n) + (cu, v_n) = (f, v_n) \to (a \nabla u, v) + (b \cdot \nabla u, v) + (cu, v) = (f, v).
\]

**Definition:** (Weak/Variational solution of BVP) Find \( u \in H^1_0 \) such that

\[
\int_\Omega a \nabla u \cdot \nabla v + b \cdot \nabla u v + c uv dx = \int_\Omega f v dx, \forall v \in H^1_0.
\]

**Terminology:** Such a function \( u \) is called a weak or variational solution of BVP. Note: The above calculation shows that a classical solution is weak solution. Conversely: If \( u \) is a weak solution and \( u \in C^2(\Omega) \) then \( u \) is a classical solution. Reversing the above calculation we find that

\[
\int_\Omega f v dx = \int_\Omega Lu v = \int_\Omega \forall v \in C^0_0
\]

or

\[
\int_\Omega (Lu - f) v dx = 0 \forall v \in C^0_0.
\]

\((Lu - f, v) = 0 \forall v \in C^0_0 \). As \( C^0_0 \) is dense in \( L^2 \) we conclude that \( Lu - f \equiv 0 \) in \( L^2 \) that is \( Lu - f = 0 \) a.e. If \( u \in C^2(\Omega) \) and \( f \in C(\Omega) \Rightarrow Lu - f \in C(\Omega) \Rightarrow Lu(x) - f(x) = 0 \) for all \( x \in \Omega \).

(If \( g \) is continuous on \( \Omega \) and \( g = 0 \) a.e then \( g = 0 \) \( \forall x \in \Omega \) Finally as \( u \in H^1_0 \cap C^2(\Omega) \), we have \((\gamma u)(x) = u(x), \forall x \in \Gamma \Rightarrow u = 0 \) on \( \Gamma \) thus \( u \) is a classical solution.

Note: A weak solution is often not regular enough to be a classical solution (e.g \( f \in L^2, \Omega \) has corners etc.).

**Theorem:** Suppose that \( a, b \) and \( c \) are smooth functions in \( \Omega \) and that \( a(x) \geq a_0 > 0 \) and that \( c(x) - \frac{1}{2} \nabla \cdot b \geq 0 \) for all \( x \in \Omega \) and \( f \in L^2 \). Then there is a unique weak solution \( u \) of BVP. That is, there is a unique \( u \in H^1_0 \) such that

\[
\int_\Omega a \nabla u \cdot \nabla v + b \cdot \nabla u v + c uv dx = \int_\Omega f v dx \forall v \in H^1_0
\]

Furthermore there is a constant \( c > 0 \) independent of \( f \) such that \( |u|_1 \leq c ||f||_{L^2} \).

**Proof:** We will use the Lax-Milgram Lemma on \( V = H^1_0 \) with norm \( | \cdot |_1 \), bilinear form

\[
a(w, v) = \int_\Omega a \nabla w \cdot \nabla v + b \cdot \nabla u v + c uv dx, w, v \in H^1_0 = V
\]

and linear functional \( L(v) = \int_\Omega f v dx \). We need to check that \( a \) is bilinear bounded and coercive, we also need to check that \( L : V \to \mathbb{R} \) is bounded.
To begin with we will need some inequalities they are

\[ \|f \cdot g\|_{L^2} \leq \|f\|_{L^\infty} \cdot \|g\|_{L^2} \]

If \( F = (f_1, \ldots, f_d) \) \( G = (g_1, \ldots, g_d) \)

\[ \left| \int_{\Omega} F \cdot G dx \right| \leq \|F\|_{L^2} \cdot \|G\|_{L^2} \]

where \( \|F\|_{L^2} = \int_{\Omega} \sum_{j=1}^{d} f_j^2 dx \)

\[ \|F \cdot G\|_{L^2} \leq \max_{1 \leq i \leq d} \|f_i\|_{L^\infty} \|G\|_{L^2} \]

\[ \|f F\|_{L^2} \leq \|f\|_{L^\infty} \|F\|_{L^2} . \]

The proof continues in the next lecture.
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Proof: We will use Lax-Milgram Lemma: If $V$ is a Hilbert space, $a: V \times V \to \mathbb{R}$ is a bounded coercive bilinear form on $V$ and $L: V \to \mathbb{R}$ is a bounded linear functional on $V$ then there is a unique $u \in V$ such that $a(u, v) = L(v) \quad \forall v \in V$ and $\|u\|_V \leq c\|L\|_{V^*} = \sup_{v \in V} \frac{|L(v)|}{\|v\|_V}$.

Let $V = H^1_0$ with norm $\| \cdot \|_1$, define

$$a(w, v) = \int_\Omega a \nabla w \cdot \nabla v + b \cdot \nabla w v + cwv dx \quad v, w \in H^1_0 = V$$

and define

$$L(v) = \int_\Omega f v dx \quad v \in H^1_0.$$

As stated we need to show: $a$ is (1) bilinear, (2) bounded and (3) coercive, we also have to check if (4) $L$ is bounded. It is easy to see that $a$ is bilinear, that takes care of criterion (1). We now show that $a$ is bounded, that is $|a(w, v)| \leq K|w|_1|v|_1$: 

$$|a(w, v)| \leq \left| \int_\Omega a \nabla w \cdot \nabla v dx \right| + \left| \int_\Omega b \cdot \nabla w v dx \right| + \left| \int cwv dx \right| \leq C \|w\|_2 \|v\|_2 + |b| \|\nabla w\|_2 \|v\|_2 + |c| \|w\|_2 \|v\|_2$$

Poincaré

$$|a(v)| \leq \|a\|_{L^\infty} \|\nabla w\|_{L^p} \|v\|_{L^p} + (\max_{1 \leq i \leq d} |b_i|) \|\nabla w\|_{L^2} \|v\|_{L^2} + |c| \|w\|_{L^2} \|v\|_{L^2}$$

Note that we have used the definition of the seminorm here

$$K = 3 \max \left\{ |a|_{L^\infty} + M \left( \max_{1 \leq i \leq d} |b_i| \|v\|_{L^\infty} \right), M^2 |c|_{L^\infty} \|v\|_{L^1} \right\}.$$

We have now shown the boundedness of $a$. We now show coercivity that is $|a(v, v)| \geq \alpha \|v\|_V^2$.

$$a(v, v) = \int_\Omega a |\nabla v|^2 + b \cdot \nabla v + c v^2 dx = \int_\Omega a |\nabla v|^2 + \frac{1}{2} \nabla b v^2 + c v^2 dx$$

Note: $\nabla \cdot (bv) = \nabla v \cdot b + b \cdot \nabla (v^2)$. Also since $v$ is zero on $\Gamma$ since $v \in H^1_0$ the divergence theorem gives us that

$$\int_\Omega \nabla \cdot (bv) dx = \int_{\gamma} b \cdot n v^2 ds = 0 \Rightarrow \int_\Omega b \cdot \nabla (v^2) dx = - \int_\Omega v^2 \nabla \cdot b dx.$$

Thus we have that

$$a(v, v) = \int_\Omega a |\nabla v|^2 + \frac{1}{2} \nabla v \cdot b + c v^2 dx = \int_\Omega a |\nabla v|^2 + (c - \frac{1}{2} \nabla \cdot b) v^2 dx$$

$$\geq \int_\Omega a |\nabla v|^2 dx \geq a_0 \int_\Omega |\nabla v|^2 dx = a_0 |v|^2_1,$$

(here we used that $c - \frac{1}{2} \nabla \cdot b \geq 0$) this means $a$ is coercive. Finally, we need to show that $L$ is bounded, that is show $\exists C > 0 : |L(v)| \leq C\|v\|_V$. We have

$$|L(v)| = |(v, f)| \leq \|v\| \|f\| \leq C\|f\| \|v\|_1$$

$$\Rightarrow \|L\|_{V^*} \leq \sup_{v \in V} \frac{|L(v)|}{\|v\|_1} \leq C\|f\|. $$
Which shows that $L$ is bounded. Now by the Lax-Milgram lemma there is a unique $w \in V = H^1_0$ such that $a(w,v) = L(v) \forall v \in V = H^1_0$ and $|w|_1 = ||w||_V \leq C||L||_V \leq K||f||$.

When $b = 0$ the bilinear form $a$ is symmetric, then the unique weak solution can be characterized as the minimizer of the energy functional $F(v) = \frac{1}{2}a(v,v) - L(v)$.

**Theorem:** (Dirichlet’s principle) Suppose that $b = 0$, $a$, $c$ are smooth in $\bar{\Omega}$ and $a(x) > a_0 > 0$ $c(x) > 0$ $x \in \Omega$ then the unique solution of BVP satisfies $F(u) \leq F(v) \forall v \in H^1_0$ where

$$F(v) = \frac{1}{2} \int_{\Omega} a|\nabla v|^2 dv - \int_{\Omega} fvdx$$

with equality only if $v = u$.

**Proof:** Theorem A.2 (in the book) shows that $F(u) \leq F(v) \forall v \in V = H^1_0$ as $u$ is a weak solution. If $w \in H^1_0$ such that $F(w) \leq F(v)$ for all $v \in H^1_0$ then by theorem A.2, $w$ is a weak solution. By uniqueness $u = w$.

**Inhomogeneous BVP:** Classical formulation: $u \in C^2$ such that $Lu = f$ in $\Omega$, $u = g$ on $\Gamma$ where $f$ and $g$ are given continuous functions.

We would like to consider this problem when $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$. Weak formulation: Find $u \in H^1$ such that $a(u,v) = L(v)$ for all $v \in H^1_0$ $\gamma u = g$ where $\gamma^1_H \rightarrow L^2(\Gamma)$ is the trace operator

$$a(u,v) = \int_{\Omega} a\nabla u \cdot \nabla v + b \cdot \nabla uv + cuvdx,$$

$$L(v) = \int_{\Omega} fvdx.$$

Call this problem BVP1.

**Theorem:** Suppose that there is an $u_0 \in H^1$ such that $\gamma u_0 = g$. If $a$, $b$, $c$ are smooth, $a(x) \geq a_0 > 0$, $c(x) - \frac{1}{4} \nabla b(x) \geq 0$ for all $x \in \Omega$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ then there is a unique weak solution of BVP1.

**Proof:** We look at the problem: find $w \in H^1_0$ such that $a(w,v) = L(v) - a(w,v) \forall v \in H^1_0$. As $a : H^1_0 \times H^1_0 \rightarrow \mathbb{R}$ is bounded and coercive (like before), $L$ is bounded, and $V \rightarrow a(u_0,v)$ is also bounded on $H^1_0$. We have that

$$|a(u_0,v)| \leq K||u_0||_1||v||_1.$$ 

By Lax-Milgram there is a unique $w \in H^1_0$ such that $a(w,v) = L(v) - a(u_0,v) \forall v \in H^1_0$. Then $u := w + u_0$ is a weak solution of BVP1.

$$a(u,v) = a(w,v) + a(u_0,v) = L(v) - a(u_0,v) + a(u_0,v) = L(v).$$

Also

$$\gamma u = \gamma w + \gamma u_0 = 0 + g = g,$$

hence $u$ is a weak solution of BVP1. Uniqueness: Suppose that $w_1$ and $w_2$ are weak solutions of BVP1. Let $u = w_1 - w_2$,

$$a(u,v) = a(w_1,v) - a(w_2,v) = L(v) - L(v) = 0 \forall v \in H^1_0.$$

$$\gamma u = \gamma w_1 - \gamma w_2 = g - g = 0,$$

hence $u \in H^1_0$. Furthermore $u$ solves $a(u,v) = 0$ for all $v \in H^1_0$. But this has a unique solution which has to be $u$, which that satisfies $|u|_1 < ||f|| = c||0|| = 0 \Rightarrow u = 0$ in $H^1 \Rightarrow w_1 = w_2$. 

19.
Neumann problem: We consider the classical formulation: Find \( u \in C^2(\Omega) \) such that
\[ Au = -\nabla \cdot (a \nabla u) + cu = f \] in \( \Omega \), \( \frac{\partial u}{\partial n} = 0 \) on \( \Gamma \), where \( \frac{\partial u}{\partial n} = n \cdot \nabla u \) where \( n \) is the unit normal of \( \Gamma \). Let \( u \in C^2(\Omega) \) be a classical solution and \( v \in C^1(\Omega) \) then
\[
\int_{\Omega} vf\,dx = \int_{\Omega} Auv\,dx = \int_{\Omega} -\nabla \cdot (a \nabla u) + cu\,dx = - \int_{\Gamma} a\nabla u \cdot nv\,ds + \int_{\Omega} a\nabla u \cdot \nabla v + cuv\,dx = \int_{\Omega} a\nabla u \cdot \nabla v + cuv\,dx \forall v \in C^1(\Omega).
\]
Here we used that the normal derivative is 0. By limit argument using that \( C^1(\Omega) \) is dense in \( H^1(\Omega) \) we set
\[
\int_{\Omega} a\nabla u \cdot \nabla v + cuv\,dx = \int_{\Omega} f\,dx \forall v \in H^1.
\]
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Neumann problem continued (Weak formulation): Find \( u \in H^1(\Omega) \) such that

\[
\int_{\Omega} a \nabla u \cdot \nabla v + cuv \, dx \quad \forall v \in H^1(\Omega),
\]

if \( u \) is a weak solution and \( u \in C^1 \), then \( u \) is a classical solution. Indeed: reversing the steps before we get

\[
\int_{\Omega} f v \, dx = \int_{\Omega} -\nabla \cdot (a \nabla u) v + cuv \, dx + \int_{\Gamma} a \frac{\partial u}{\partial n} v \, ds \quad \forall v \in H^1.
\]

Let first \( v \in C^1_0 \subset H^1 \Rightarrow \int_{\Omega} f v \, dx = \int_{\Omega} -\nabla \cdot (a \nabla u) v + cuv \, dx \quad \forall v \in C^1_0 \Rightarrow \int(\mathcal{L} u - f) v \, dx = 0 \quad \forall v \in C^1_0.
\]

Since \( C^1_0 \) is dense in \( L^2 \), we get \( \mathcal{L} u = f \) a.e. If \( u \in C^2(\overline{\Omega}), f \in C(\overline{\Omega}) \Rightarrow \mathcal{L} u(x) = f(x) \) in \( \Omega \Rightarrow \int_{\Gamma} a \frac{\partial u}{\partial n} v \, ds = 0 \quad \forall v \in H^1 \Rightarrow \frac{\partial u}{\partial n} = 0 \) on \( \Gamma \).

**Theorem:** Let \( a, b, c \) be smooth in \( \overline{\Omega} \), \( a(x) \geq a_0 > 0 \quad \forall x \in \Omega, c(x) \geq c_0 > 0 \) and \( f \in L^2 \). Then the Neumann boundary value problem has a unique weak solution.

**Proof:** Let

\[
a(w, v) = \int_{\Omega} a \nabla w \nabla v + cvw \, dx, \quad v, w \in H^1
\]

and let

\[
L(v) = \int_{\Omega} f v \, dx \quad v \in H^1.
\]

To Show: There is a unique \( u \in H^1 : a(u, v) = L(v) \forall v \in H^1 \). To show: \( a \) is bounded and coercive (\( a \) is clearly symmetric and bilinear!).

Bounded:

\[
|a(w, v)| \leq \int_{\Omega} a \nabla w \cdot \nabla v \, dx + \int_{\Omega} cvw \, dx \leq ||a||_{L^\infty} ||\nabla w||_L ||\nabla v||_L + ||c||_{L^\infty} ||v||_L \leq ||a||_{L^\infty} ||\nabla w||_L ||\nabla v||_L + ||c||_{L^\infty} ||v||_L \leq k ||w||_1 ||v||_1,
\]

and

\[
||a||_{L^\infty} (||w|| + ||\nabla w||) (||c|| + ||\nabla v||) + ||c||_{L^\infty} (||w|| + ||\nabla w||) (||v|| + ||\nabla v||) = k ||w||_1 ||v||_1,
\]

and where \( k = ||a||_{L^\infty} + ||c||_{L^\infty} \).

Coercive:

By the Riesz representation theorem (or more generally by Lax-Milgram)

\[
\exists u \in H^1 : a(u, v) = L(v) \forall v \in H^1
\]