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MVE550 Stochastic Processes and Bayesian Inference

Exam April 13, 2022, 8:30 - 12:30
Examiner: Petter Mostad, phone 031-772-3579, visits exam at 9:30 and 11:30
Allowed aids: Chalmers-approved calculator
Total number of points: 30. At least 12 points are needed to pass.
See appendix for some information about some probability distributions

1. (4 points) Assume X has a Negative Binimial distribution with parameters r and p, written $X \sim \text{Negative-Binomial}(r, p)$. See the appendix for the definition of the Negative Binomial distribution. We assume r is fixed and known, while p is an unknown parameter.
   (a) Prove that for the parameter p, the Beta[1] family of distributions is a conjugate family.
   (b) Assume that $r = 4$ and that we use a uniform prior on p. Assume we make three independent observations 4, 2, 3 of X. What is the posterior distribution for p given these observations?

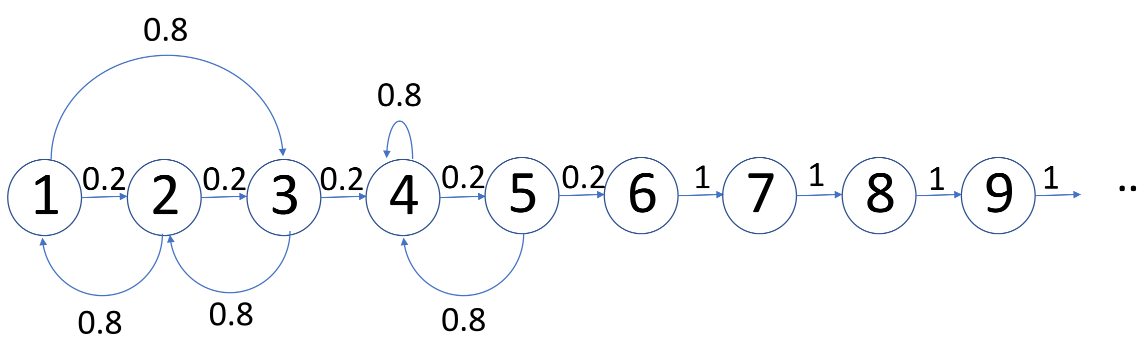

Figure 1: The transition graph for the Markov chain of question 2.

[1] There was an error in the original exam: It said "Binomial" instead of "Beta"

2. (6 points) The transition graph of a discrete-time infinite state space Markov chain is given in Figure 1.

   (a) We would like to compute the probability that, if the chain starts in state 1 at time step zero, it is in state 4 at time step 9, and then continues to[2] state 6 at time step 14. Write down a way to compute this. You may express your answer as an expression consisting of finite-dimensional matrices and vectors containing numbers, and operations on these such as multiplication, addition, subtraction, inversion, and exponentiation. Alternatively, you may directly write an R command which would compute the result.

   (b) Which are the communication classes of the Markov chain? Which of these are open and which are closed? Which states are transient, and which are recurrent?

   (c) Write down a way to compute the expected number of steps until the chain reaches state 6, if it starts at state 1. You may express your answer as an expression consisting of finite-dimensional matrices and vectors containing numbers, and operations on these such as multiplication, addition, subtraction, inversion, and exponentiation. Alternatively, you may directly write an R command which would compute the result.

3. (6 points) A Branching process has offspring process where the probabilities of zero, one, or two offspring are $1 - a$, 0, and a, respectively, where $0 < a < 1$.

   (a) For what values of a is the process critical, supercritical, and subcritical?

   (b) What is the expected size of the n'th generation?

   (c) What is the probability of exctinction, for the different possible values of a?

   (d) Change the offspring process so that to the offspring described above are independently added offspring according to a Poisson process with parameter 2. If we set $a = 2/3$, describe how to compute the probability of extinction. Your description may include a numerical computation that you do not actually perform, only outline.

4. (4 points) Let N_t be a Poisson process with parameter λ. Let $X_1, X_2, \ldots$ be the waiting times until the first arrival, between the first and second arrivals, and so on.

   (a) Compute $E(N_2 N_4)$.

   (b) Compute $E(X_2 X_4)$.

   (c) Select uniformly at random one of the arrivals that have occurred before time 4. Compute its expected arrival time.

5. (6 points) A computer system can operate in its standby state S_0 or in one of k different active states $S_1, S_2, \ldots, S_k$, where k is some positive integer. From S_0 it changes to S_i at the rate $4/i$, and from S_i it changes back to S_0 at the rate $i/4$. No other transitions are possible.

[2] The formulation was slightly different, and possible to misunderstand, in the original exam.

(a) Write down the generator matrix Q for the continuous-time Markov chain above when $k = 4$.

(b) The probability that the system is in state S_1 at time t, is it independent of the state the system is at time zero, when $t \to \infty$? Prove or disprove.

(c) Compute the long-term probability that the system is in state S_0, assuming that $k = 4$.

(d) Is the Markov chain time reversible or not? Explain your answer.

6. (4 points)

(a) Write down the definition of geometric Brownian motion.

(b) If G_t denotes geometric Brownian motion, compute $E(G_t)$.

Appendix: Some probability distributions

The Beta distribution

If $x \in [0, 1]$ has a Beta distribution with parameters with $\alpha > 0$ and $\beta > 0$ then the density is

$$\pi(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

We write $x \mid \alpha, \beta \sim \text{Beta}(\alpha, \beta)$ and $\pi(x \mid \alpha, \beta) = \text{Beta}(x; \alpha, \beta)$.

The Beta-Binomial distribution

If $x \in \{0, 1, 2, \ldots, n\}$ has a Beta-Binomial distribution, with n a positive integer and parameters $\alpha > 0$ and $\beta > 0$, then the probability mass function is

$$\pi(x \mid n, \alpha, \beta) = \binom{n}{x} \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}.$$

We write $x \mid n, \alpha, \beta \sim \text{Beta-Binomial}(n, \alpha, \beta)$ and $\pi(x \mid n, \alpha, \beta) = \text{Beta-Binomial}(x; n, \alpha, \beta)$.

The Binomial distribution

If $x \in \{0, 1, 2, \ldots, n\}$ has a Binomial distribution, with n a positive integer and $0 \leq p \leq 1$, then the probability mass function is

$$\pi(x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}.$$

We write $x \mid n, p \sim \text{Binomial}(n, p)$ and $\pi(x \mid n, p) = \text{Binomial}(x; n, p)$.

The Dirichlet distribution

If $x = (x_1, x_2, \ldots, x_n)$ has a Dirichlet distribution, with $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$ and with parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > 0, \ldots, \alpha_n > 0$, then the density function is

$$\pi(x \mid \alpha) = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} p_1^{\alpha_1-1} p_2^{\alpha_2-1} \cdots p_n^{\alpha_n-1}.$$

We write $x \mid \alpha \sim \text{Dirichlet}(\alpha)$ and $\pi(x \mid \alpha) = \text{Dirichlet}(x; \alpha)$.

The Exponential distribution

If $x \geq 0$ has an Exponential distribution with parameter $\lambda > 0$, then the density is

$$\pi(x \mid \lambda) = \lambda \exp(-\lambda x)$$

We write $x \mid \lambda \sim \text{Exponential}(\lambda)$ and $\pi(x \mid \lambda) = \text{Exponential}(x; \lambda)$. The expectation is $1/\lambda$ and the variance is $1/\lambda^2$.

The Gamma distribution

If $x > 0$ has a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ then the density is

$$\pi(x \mid \alpha\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x).$$

We write $x \mid \alpha, \beta \sim \text{Gamma}(\alpha, \beta)$ and $\pi(x \mid \alpha, \beta) = \text{Gamma}(x; \alpha, \beta)$.

The Geometric distribution

If $x \in \{1, 2, 3, \dots\}$ has a Geometric distribution with parameter $p \in (0, 1)$, the probability mass function is

$$\pi(x \mid p) = p(1-p)^{x-1}$$

We write $x \mid p \sim \text{Geometric}(p)$ and $\pi(x \mid p) = \text{Geometric}(x; p)$. The expectation is $1/p$ and the variance $(1-p)/p^2$.

The Negative Binomial distribution

A stochastic variable x taking on as possible values any nonnegative integer has a Negative Binomial distribution if its probability mass function is given by

$$\pi(x \mid r, p) = \binom{x+r-1}{x} \cdot (1-p)^x p^r = \frac{\Gamma(x+r)}{\Gamma(x+1)\Gamma(r)}(1-p)^x p^r$$

where $r > 0$ and $p \in (0, 1)$ are parameters.

The Normal distribution

If the real x has a Normal distribution with parameters μ and σ^2, its density is given by

$$\pi(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).$$

We write $x \mid \mu, \sigma^2 \sim \text{Normal}(\mu, \sigma^2)$ and $\pi(x \mid \mu, \sigma^2) = \text{Normal}(x; \mu, \sigma^2)$.

The Poisson distribution

If $x \in \{0, 1, 2, \dots\}$ has Poisson distribution with parameter $\lambda > 0$ then the probability mass function is

$$e^{-\lambda}\frac{\lambda^x}{x!}.$$

We write $x \mid \lambda \sim \text{Poisson}(\lambda)$ and $\pi(x \mid \lambda) = \text{Poisson}(x; \lambda)$. The Poisson distribution has expectation λ and variance λ.

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Suggested solutions for
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1. (a) Assume that we use the prior $p \sim \text{Beta}(\alpha, \beta)$ so that

$$\pi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}.$$

To prove conjugacy we need to prove that the posterior is then also a Beta distribution. We get

$$\begin{aligned}
\pi(p \mid x) &\propto_p \pi(x \mid p)\pi(p) \\
&\propto_p \binom{x+r-1}{x} \cdot (1-p)^x p^r p^{\alpha-1}(1-p)^{\beta-1} \\
&\propto_p p^{\alpha+r-1}(1-p)^{\beta+x-1} \\
&\propto_p \text{Beta}(p; \alpha+r, \beta+x)
\end{aligned}$$

which shows that the posterior is a Beta distribution.

(b) Note that the Uniform distribution on $[0,1]$ is the same as a Beta$(1,1)$ distribution. So using $\alpha = 1$ and $\beta = 1$ and repeated Bayesian update of the parameter, we get the posterior

$$\pi(p \mid \text{data}) = \text{Beta}(1+4+4+4, 1+4+2+3) = \text{Beta}(13, 10).$$

2. (a) First of all, for the purposes of this question, we may change the Markov chain by removing states above 7 and making state 7 into an absorbing state: This is OK because once the chain reaches state 7 it will never again return to lower states.

Let P be the transition matrix of this simplified chain:

$$P = \begin{bmatrix}
0 & 0.2 & 0.8 & 0 & 0 & 0 & 0 \\
0.8 & 0 & 0.2 & 0 & 0 & 0 & 0 \\
0 & 0.8 & 0 & 0.2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.8 & 0.2 & 0 & 0 \\
0 & 0 & 0 & 0.8 & 0 & 0.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Then the required probability is $(P^9)_{14}(P^5)_{46}$. This can be computed in R with

```
matrixpower(P, 9)[1,4]*matrixpower(P, 5)[4,6]
```

provided `matrixpower` computes the power of a matrix. We might also write this out as a matrix product

$$v_1 PPPPPPPPPv_4^t v_4 PPPPPv_6^t$$

where

$$\begin{aligned} v_1 &= \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \\ v_4 &= \begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0 & 0\end{bmatrix} \\ v_6 &= \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}. \end{aligned}$$

(b) The communication classes are $\{1, 2, 3\}$, $\{4, 5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, ... They are all open, and all states are transient, as for any states there is a nonzero probability that one will never return to this state.

(c) Making state 6 absorbing, we get

$$Q = \begin{bmatrix} 0 & 0.2 & 0.8 & 0 & 0 \\ 0.8 & 0 & 0.2 & 0 & 0 \\ 0 & 0.8 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0.8 & 0 \end{bmatrix}$$

and the answer is given by the sum of the first row of the fundamental matrix $F = (I - Q)^{-1}$. In R we might write

```
sum(solve(diag(5)-Q)[1,])
```

A mathematical way to write this might be

$$[1 \quad 0 \quad 0 \quad 0 \quad 0](I - Q)^{-1} \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}.$$

3. (a) If X is a random variable with the offspring distribution have that $\mu = \mathrm{E}[X] = 0 \cdot (1 - a) + 1 \cdot 0 + 2 \cdot a = 2a$. We compare this number with 1, and conclude that the process is critical if $a = 1/2$, supercritical if $a > 1/2$, and subcritical if $a < 1/2$.

(b) The expected size is

$$\mu^n = (2a)^n.$$

(c) We first find the probability generating function:

$$G(s) = (1-a)s^0 + 0s^1 + as^2 = 1 - a + as^2.$$

Then we know that the extinction probability is the smallest positive root of $G(s) = s$. We get (using that we know $s = 1$ is always a root)

$$\begin{aligned} 1 - a + as^2 &= s \\ as^2 - s + 1 - a &= 0 \\ (s-1)(as + a - 1) &= 0. \end{aligned}$$

As $as + a - 1 = 0$ solves to

$$s = \frac{1-a}{a} = \frac{1}{a} - 1,$$

we see that if $a \leq 1/2$ the extinction probability is 1, while if $a > 1/2$ the extinction probability is $1/a - 1$.

(d) Let X be the original number of offspring and Y be the offspring from the Poisson process. Then

$$\begin{aligned} G(s) &= \mathrm{E}\left[s^{X+Y}\right] = \mathrm{E}\left[s^X\right]\mathrm{E}\left[s^Y\right] \\ &= (1 - \frac{2}{3} + \frac{2}{3}s^2)\sum_{k=0}^{\infty} s^k e^{-2}\frac{2^k}{k!} \\ &= \left(\frac{1}{3} + \frac{2}{3}s^2\right)e^{-2}\sum_{k=0}^{\infty}\frac{(2s)^k}{k!} \\ &= \frac{1+2s^2}{3}e^2 e^{2s} = \frac{1}{3}(1+2s^2)e^{2(s-1)} \end{aligned}$$

We want to find the smallest positive root of $G(s) = s$, so we study

$$f(s) = \frac{1}{3}(1+2s^2)e^{2(s-1)} - s.$$

As $\mathrm{E}[X+Y] = \frac{4}{3} + 2 > 1$ and the process is supercritical we know $f(s)$ will have a root in the interval $(0,1)$. In R one may use for example the function `uniroot`, on the function $f(s)$, making sure to aviod the root $s = 1$.

4. (a) We can compute

$$\begin{aligned} \mathrm{E}[N_2 N_4] &= \mathrm{E}[N_2(N_4 - N_2 + N_2)] \\ &= \mathrm{E}\left[N_2(N_4 - N_2) + N_2^2\right] \\ &= \mathrm{E}[N_2]\,\mathrm{E}[N_4 - N_2] + \mathrm{E}\left[N_2^2\right] \\ &= \mathrm{E}[N_2]\,\mathrm{E}[N_2] + \mathrm{Var}[N_2] + \mathrm{E}[N_2]^2 \end{aligned}$$

As $N_2 \sim \text{Poisson}(2\lambda)$ we have from the appendix that $\mathrm{E}[N_2] = 2\lambda$ and $\mathrm{Var}[N_2] = 2\lambda$, so the answer becomes

$$\mathrm{E}[N_2 N_4] = (2\lambda)^2 + 2\lambda + (2\lambda)^2 = 8\lambda^2 + 2\lambda.$$

(b) As $X_2 \sim \text{Exponential}(\lambda)$ and independently $X_4 \sim \text{Exponential}(\lambda)$ we get $\mathrm{E}[X_2X_4] = \mathrm{E}[X_2]\,\mathrm{E}[X_4] = \frac{1}{\lambda^2}$.

(c) A uniformly selected arrival has a uniform distribution on $[0,4]$. Its expected value is thus $4/2 = 2$.

5. (a) We get

$$Q = \begin{bmatrix} -25/3 & 4 & 2 & 4/3 & 1 \\ 1/4 & -1/4 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 \\ 3/4 & 0 & 0 & -3/4 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

(b) As the Markov chain is a finite irreducible continuous-time Markov chain it has a unique stationary distribution which is the limiting distribution. This implies that as $t \to \infty$ the state it is in becomes independent of the starting state.

(c) We solve the system

$$v \begin{bmatrix} 1 & 4 & 2 & 4/3 & 1 \\ 1 & -1/4 & 0 & 0 & 0 \\ 1 & 0 & -1/2 & 0 & 0 \\ 1 & 0 & 0 & -3/4 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

were we write $v = [v_0, v_1, v_2, v_3, v_4]$. We quickly get the equations

$$\begin{aligned} v_1 &= 16v_0 \\ v_2 &= 4v_0 \\ v_3 &= \frac{16}{9}v_0 \\ v_4 &= v_0 \end{aligned}$$

Together with the equation $v_0 + v_1 + v_2 + v_3 + v_4 = 1$ we get $v_0 = \frac{9}{214}$, so the answer is $\frac{9}{214}$.

(d) We see directly that the transition rate graph is a star in this case, i.e., a tree, so the Markov chain is necessarily time reversible.

6. (a) A process G_t is geometric Brownian motion if there are parameters G_0, μ, and σ so that

$$G_t = G_0 e^{t\mu + \sigma B_t}$$

where B_t is Brownian motion.

(b) We get

$$
\begin{aligned}
E(G_t) &= E(G_0 e^{t\mu+\sigma B_t}) \\
&= G_0 e^{t\mu} E(e^{\sigma B_t}) \\
&= G_0 e^{t\mu} \int_{-\infty}^{\infty} e^{\sigma s} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} s^2\right) ds \\
&= G_0 e^{t\mu} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2t}(s^2 - 2\sigma t s)\right) ds \\
&= G_0 e^{t\mu} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2t}(s-\sigma t)^2 + t\frac{\sigma^2}{2}\right) ds \\
&= G_0 e^{t\mu} e^{t\sigma^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(s-\sigma t)^2\right) ds \\
&= G_0 e^{t(\mu+\sigma^2/2)}
\end{aligned}
$$