Tentamen
MVE505 Diskret Matematik TM1

2024-10-11 kl. 14.00–18.00

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Hjälpmedel: Inga

För godkänt på tentan krävs 45 poäng, inklusive eventuella bonuspoäng erhållna från inlämningsuppgifterna under VT-2024. Preliminärt så krävs 65 poäng för betyget 4 och 85 poäng för betyget 5. Dessa gränser kan minskas men inte höjas i efterhand.

Lösningar läggs ut på kursens Canvassida direkt efter tentan. Tentan rättas och bedöms anonymt. Resultatet meddelas i Ladok senast den 1 november. Granskning ordnas därefter av kursansvarig.

OBS! Alla stegen i dina resonemang måste motiveras väl i skrift och alla beräkningar visas. Det är i huvudsak tillvägagångssätten och motiveringarna som ger poäng, inte svaren.

I de uppgifter som består av fler olika delar går det alltid att lösa de enskilda delarna oberoende av varandra, även om man kan ibland spara räknetid genom att lösa deluppgifterna sekventiellt.

Om du i en lösning av någon av uppgifterna 1-7 åberopar en sats från kurslitteraturen så behöver du *inte* inkludera ett bevis av satsen.

I Uppgift 1 behöver du *inte* ange svaren som explicita decimaltal.

Jag bifogar 3 exemplar av Figure 6 så att ni har extra kladdpapper.

Var god vänd!

Uppgifterna

1. You have 100 bananas and 10 monkeys. The monkeys have different names.

(a) How many possibilities are there for the entire sequence of names of the monkeys getting the bananas (each banana is given to one monkey) ? (3p)

(b) Same question as in (a), but assuming you want to feed each monkey exactly 10 times ? (3p)

(c) How many possibilities are there if you only care about how many bananas each monkey gets, and each monkey is to be given at least one banana ? (3p)

(d) Same question as in (c), but assuming no monkey is to be fed more than 30 times ? (3p)

(e) Call the monkeys A-J. Suppose the recipient of each banana is chosen uniformly at random, and independent of all other bananas. What is the probability that A and B together receive a total of exactly 3 bananas ? (3p)

2. Compute an explicit formula for the numbers $(a_n)_{n=0}^{\infty}$ which satisfy the recursion (12p)

$$a_0 = 1, \;\; a_1 = 1, \quad a_n = 8a_{n-1} - 16a_{n-2} + 4^{n+1} - 9n, \;\; \forall\, n \geq 2.$$

(6.5p)

3. (a) i. For which $b \in \mathbb{Z}$ does the following congruence have a solution ?

$$581x \equiv b \pmod{2177}.$$

ii. Determine the general solution to the congruence when $b = 84$, along with the largest negative solution.

(b) Write down the general solution to the Diophantine equation (2.5p)

$$2177x + 581y = 84,$$

along with the solution which minimizes $|x| + |y|$.

4. (a) Use the repeated squaring algorithm to compute $2^{41} \pmod{83}$. (5p)

OBS! *Zero* points for doing the computation some other way, the point of the exercise is to show you understand how the algorithm works. By the way, you may have use for the fact that $49^2 = 2401 = 83 \cdot 29 - 6$.

(b) How many primitive roots are there modulo 83 ? (2p)

(c) Is 2 a primitive root modulo 83 ? Motivate your answer ! (2p)

5. Compute the general solution, plus the positive solution nearest to 2000, to the following system of congruences: (8p)

$$4x \equiv 1 \pmod{7}, \quad 3x \equiv 1 \pmod{10}, \quad 2x \equiv 3 \pmod{11}.$$

6. Let G be the network in Figure 6, let G^* be the underlying undirected, but still weighted graph and let G^{**} be the underlying undirected and unweighted graph.

(a) Determine, with proof, $\chi(G^{**})$ (OBS! You can appeal to theorems in the lecture notes without proof). (3p)

(b) Starting from s, apply Dijkstra's algorithm to determine a shortest path from s to t in G^*. Indicate clearly which edge is chosen and which label is set at each step and write down the shortest path obtained. (5p)

(c) Implement the Ford-Fulkerson algorithm to determine a maximum flow from s to t in the network G and a corresponding minumum cut. Write clearly which f-augmenting path you choose at each step. Draw the final flow in full and indicate the corresponding minimum cut. (7p)

7. Apply the Gale-Shapley algorithm to determine a stable matching for the data in Figure 7. OBS! that, by convention, X is the set of proposers. Write clearly the proposals made, rejections issued and the state of the strings at each step. Indicate also clearly the final stable matching. (8p)

8. State and prove the Fundamental Theorem of Arithmetic. (12p)

9. (a) Let G be a connected graph and M a matching in G. Define what is meant by an *M-augmenting (a.k.a. M-alternating)* path in G. (2p)

(b) Formulate and prove Hall's theorem for a bipartite graph $G = (X, Y, E)$. (10p)

Go n'eirí an bóthar libh!

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1. (a) 10 choices per banana $\Rightarrow 10^{100}$ possible sequences.

(b) Call the monkeys A-J. The number of possibilities equals the number of 100-letter words consisting of 10 copies of each letter. By the Multinomial Theorem, there are $\frac{100!}{(10!)^{10}}$ possible words.

(c) 100 indistinguishable balls (the bananas) are to be distributed amongst 10 distinguishable bins (the monkeys). Since each bin must get at least one ball, we first place one ball in each bin and thus have 90 balls which can be distributed freely. By Proposition 1.9, the number of ways this can be done is $\binom{90+10-1}{10-1} = \binom{99}{9}$.

(d) Let X be the set of all possible distributions in (c). For each $i = 1, \dots, 10$, let A_i be the subset consisting of those distributions where bin number i gets more than 30 (i.e.: at least 31) balls. Since $4 \cdot 31 > 100$, the intersection of any four A_i will be empty. Hence, by Inclusion-Exclusion and symmetry, the number of distributions satisfying our requirements is

$$|X| - 10|A_1| + \binom{10}{2}|A_1 \cap A_2| - \binom{10}{3}|A_1 \cap A_2 \cap A_3|. \tag{1}$$

First consider $|A_1|$. Place 31 balls in bin 1 and 1 ball in each other bin. The remaining 60 balls can be distributed freely, giving $|A_1| = \binom{60+10-1}{10-1} = \binom{69}{9}$.
Next, consider $|A_1 \cap A_2|$. Place 31 balls in each of bins 1 and 2, and 1 ball in every other bin. The remaining 30 balls can be distributed freely, so $|A_1 \cap A_2| = \binom{30+10-1}{10-1} = \binom{39}{9}$.
Finally, $|A_1 \cap A_2 \cap A_3| = \binom{9}{9} = 1$, since if we place 31 balls in each of bins 1,2,3 and 1 ball in every other bin, thre are no balls left to distribute.
Substituting everything into (1) gives our final answer:

$$\binom{99}{9} - 10 \cdot \binom{69}{9} + \binom{10}{2} \cdot \binom{39}{9} - \binom{10}{3}.$$

(e) Let X be the number of bananas ending up with A or B. Then X is a stochastic variable with a binomial distribution $X \in \text{Bin}(n, p)$, where $n = 100$ and $p = 2/10 = 0.2$. We have the general formula

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Here $k = 3$ and thus

$$\mathbb{P}(X = 3) = \binom{100}{3} (0.2)^3 (0.8)^{97}.$$

2. *Step 1:* The characteristic equation is $x^2 = 8x - 16$, which has the repeated roots $x_1 = x_2 = 4$. Hence the general solution of the homogeneous equation is

$$a_{h,n} = (C_1 + C_2\, n) \cdot 4^n.$$

Step 2: Since 4 is a root of multiplicity two of the characteristic equation, whereas 1 is not a root at all, our choice of a particular solution should look like

$$a_{p,n} = a_{p_1,n} + a_{p_2,n},$$

where

$$a_{p_1,n} = C_3 \cdot n^2 \cdot 4^n, \quad a_{p_2,n} = C_4 n + C_5.$$

Inserting into the recurrence gives, firstly,

$$C_3 \cdot n^2 \cdot 4^n = 8C_3 \cdot (n-1)^2 \cdot 4^{n-1} - 16C_3 \cdot (n-2)^2 \cdot 4^{n-2} + 4^{n+1}.$$

The coefficients of $n^2 4^n$ and of $n4^n$ will cancel exactly. Equating coefficients of 4^n gives

$$0 = 32C_3 - 64C_3 + 64 \Rightarrow C_3 = 2.$$

Secondly,

$$C_4 n + C_5 = 8[C_4(n-1) + C_5] - 16[C_4(n-2) + C_5] - 9n.$$

Equating coefficients of n yields

$$C_4 = 8C_4 - 16C_4 - 9 \Rightarrow C_4 = -1.$$

Finally, equating constant coefficients yields

$$C_5 = 8(-C_4 + C_5) - 16(-2C_4 + C_5) \overset{C_4=-1}{\Rightarrow} C_5 = -\frac{8}{3}.$$

Step 3: Hence, the general solution of the recurrence is

$$a_n = a_{h,n} + a_{p_1,n} + a_{p_2,n} = (C_1 + C_2\, n) \cdot 4^n + 2n^2 4^n - n - \frac{8}{3}.$$

Insert the initial conditions:

$$n = 0: \quad a_0 = 1 = C_1 - \frac{8}{3} \Rightarrow C_1 = \frac{11}{3},$$
$$n = 1: \quad a_1 = 1 = 4(C_1 + C_2) + 8 - 1 - \frac{8}{3} \overset{C_1=11/3}{\Rightarrow} C_2 = -\frac{9}{2}.$$

Hence,

$$a_n = \left(\frac{11}{3} - \frac{9n}{2} + 2n^2\right) \cdot 4^n - n - \frac{8}{3}.$$

3. (a) First run Euclid's algorithm forwards to find GCD(2177, 581):

$$\begin{aligned} 2177 &= 3 \cdot 581 + 434, \\ 581 &= 1 \cdot 434 + 147, \\ 434 &= 2 \cdot 147 + 140, \\ 147 &= 1 \cdot 140 + 7, \\ 140 &= 20 \cdot 7 + 0. \end{aligned}$$

So GCD(2177, 581) = 7, which means the congruence has a solution if and only if b is a multiple of 7. In that case, the general solution is

$$x \equiv \left(\frac{581}{7}\right)^{-1} \left(\frac{b}{7}\right) \pmod{\tfrac{2177}{7}}.$$

For $b = 84$ this reduces to

$$x \equiv 83^{-1} \cdot 12 \pmod{311}. \tag{2}$$

To find the inverse, we run Euclid backwards:

$$\begin{aligned} 7 &= 147 - 140 \\ &= 147 - (434 - 2 \cdot 147) \\ &= 3 \cdot 147 - 434 \\ &= 3(581 - 434) - 434 \\ &= 3 \cdot 581 - 4 \cdot 434 \\ &= 3 \cdot 581 - 4(2177 - 3 \cdot 581) \\ \Rightarrow 7 &= -4 \cdot 2177 + 15 \cdot 581 \qquad (3) \\ \Rightarrow 1 &= -4 \cdot 311 + 15 \cdot 83 \\ \Rightarrow 83^{-1} &\equiv 15 \pmod{311}. \end{aligned}$$

Substituting into (2) gives the general solution $x \equiv 15 \cdot 12 \equiv 180 \pmod{311}$. The largest negative solution will be $x = 180 - 311 = -131$.

(b) The general solution has the form

$$x = mx_0 + \left(\frac{b}{d}\right) n, \quad y = my_0 - \left(\frac{a}{d}\right) n, \quad n \in \mathbb{Z}. \tag{4}$$

Here $a = 2177$, $b = 581$, $d = 7$ and $m = c/d = 12$. From (3) we have $(x_0, y_0) = (-4, 15)$. Inserting everything into (4) gives

$$x = -48 + 83n, \quad y = 180 - 311n, \quad n \in \mathbb{Z}.$$

It is then easy to check that the solution minimizing $|x| + |y|$ is that for $n = 1$, i.e.: $x = 35$, $y = -131$.

4. (a) First, we write the power in base 2:

$$\begin{aligned} 41 &= 2 \cdot 20 + 1, \\ 20 &= 2 \cdot 10 + 0, \\ 10 &= 2 \cdot 5 + 0, \\ 5 &= 2 \cdot 2 + 1, \\ 2 &= 2 \cdot 1 + 0, \\ 1 &= 2 \cdot 0 + 1, \end{aligned}$$

which implies that

$$(41)_2 = 101001. \tag{5}$$

Secondly, we compute in turn

$$\begin{aligned} x_0 = 2^{2^0} = 2^1 &\equiv 2 \pmod{83}, \\ x_1 \equiv 2^2 &\equiv 4, \\ x_2 \equiv 4^2 &\equiv 16, \\ x_3 \equiv 16^2 \equiv 256 &\equiv 7, \\ x_4 \equiv 7^2 &\equiv 49, \\ x_5 \equiv 49^2 \equiv 2401 &\overset{\text{tipset}}{\equiv} -6. \end{aligned}$$

Finally, from (5) it follows that, modulo 83,

$$2^{41} = 2^{2^5+2^3+2^0} \equiv x_5 x_3 x_0 \equiv (-6)(7)(2) \equiv -84 \equiv -1.$$

(b) There are $\phi(\phi(83))$ primitive roots. Now 83 is a prime, so $\phi(83) = 82 = 2 \cdot 41$. Thus, $\phi(82) = \phi(2)\phi(41) = (2-1)(41-1) = 40$. So there are 40 primitive roots modulo 83.

(c) By Fermat's theorem, $2^{82} \equiv 1 \pmod{83}$, and for 2 to be a primitive root we cannot have $2^k \equiv 1 \pmod{83}$ for any $1 \le k < 82$. A priori, the only possibilities for k are the divisors of $82 = 2 \cdot 41$, thus $k \in \{1, 2, 41\}$. Now $2^1 \equiv 2$ and $2^2 \equiv 4$. By part (a), we have $2^{41} \equiv -1$. Thus no value of k works and we conclude that 2 is indeed a primitive root modulo 83.

5. First some editing:

$$\begin{aligned} 4x \equiv 1 \pmod 7 &\Rightarrow x \equiv 4^{-1} \cdot 1 \equiv 2 \cdot 1 \equiv 2 \pmod 7, \\ 3x \equiv 1 \pmod{10} &\Rightarrow x \equiv 3^{-1} \cdot 1 \equiv (-3) \cdot 1 \equiv -3 \pmod{10}, \\ 2x \equiv 3 \pmod{11} &\Rightarrow x \equiv 2^{-1} \cdot 3 \equiv 6 \cdot 3 \equiv 7 \equiv -4 \pmod{11}. \end{aligned}$$

Thus, by eq. (11.3) in the lecture notes, the general solution is

$$x \equiv 2 \cdot b_1 \cdot 10 \cdot 11 - 3 \cdot b_2 \cdot 7 \cdot 11 - 4 \cdot b_3 \cdot 7 \cdot 10 \pmod{7 \cdot 10 \cdot 11}, \tag{6}$$

where

$$\begin{aligned} b_1 \equiv (10 \cdot 11)^{-1} \equiv (3 \cdot 4)^{-1} \equiv 12^{-1} \equiv 5^{-1} &\equiv 3 \pmod 7, \\ b_2 \equiv (7 \cdot 11)^{-1} \equiv (7 \cdot 1)^{-1} \equiv 7^{-1} &\equiv 3 \pmod{10}, \\ b_3 \equiv (7 \cdot 10)^{-1} \equiv 70^{-1} \equiv 4^{-1} &\equiv 3 \pmod{11}. \end{aligned}$$

We thus choose $b_1 = b_2 = b_3 = 3$ and insert into (6) to get

$$\begin{aligned} x &\equiv 3 \cdot (2 \cdot 10 \cdot 11 - 3 \cdot 7 \cdot 11 - 4 \cdot 7 \cdot 10) \\ &\equiv 3(220 - 231 - 270) \equiv 3(-291) \equiv -873 \equiv -103 \equiv 667 \pmod{770}. \end{aligned}$$

So the general solution is $x \equiv 667 \pmod{770}$, and the solution closest to 2000 will be $x = 667 + 2 \cdot 770 = 2207$. We do a sanity check:

$667 - 2 = 665 = 7 \cdot 95$, ok

$667 - (-3) = 670 = 10 \cdot 67$, ok

$667 - (-4) = 671 = 11 \cdot 61$, ok !

6. (a) The graph is plane so $\chi(G^{**}) \leq 4$. There is a W_5 centered at b with spokes out to s, a, e, h, c. This part already requires 4 colors, thus $\chi(G^{**}) = 4$.

It's not necessary, but if you want an explicit 4-coloring, you can for example color greedily in alphabetical order. This will yield the following coloring:

Color 1: a, c, f.

Color 2: b, d, g, i.

Color 3: e, s, t.

Color 4: h.

(b) The algorithm could proceed as follows (there are other possibilities, for example the first three steps are interchangeable):

Step	Edge chosen	Label set
1	$\{s, a\}$	$l(a) := 10$
2	$\{s, b\}$	$l(b) := 10$
3	$\{s, c\}$	$l(c) := 10$
4	$\{c, h\}$	$l(h) := 13$
5	$\{a, d\}$	$l(d) := 14$
6	$\{b, e\}$	$l(e) := 14$
7	$\{c, i\}$	$l(i) := 15$
8	$\{e, f\}$	$l(f) := 16$
9	$\{f, g\}$	$l(g) := 18$
10	$\{f, t\}$	$l(t) := 21$

Reading backwards from t gives the shortest path $s \to a \to d \to f \to t$, of length 21. Note that, if one had made different choices at various steps, one might have instead obtained the path $s \to c \to h \to t$, also of length 21.

(c) The algorithm could proceed as above (there are other alternatives).

Step	f-augmenting path	Increase in flow strength
1	$s \to a \to d \to f \to t$	3
2	$s \to a \to e \to g \to t$	5
3	$s \to b \to h \to t$	6
4	$s \to b \to e \to h \to t$	2
5	$s \to b \to e \to g \to t$	1
6	$s \to c \to i \to t$	5
7	$s \to c \to h \to g \rightsquigarrow e \to f \to t$	2
Total flow strength		24

Note that, in Step 7, we reduce the flow along the arc (e, g). The final flow is illustrated in Figure L.6(c). The set of nodes that can now be reached from s via an augmenting path is $S = \{s, a, b, c, d, e, g, h\}$. Set $T = V \setminus S = \{f, i, t\}$. Then we have

$$c(S, T) = c(d, f) + c(e, f) + c(g, t) + c(h, t) + c(c, i) = 3 + 2 + 6 + 8 + 5 = 24 = |f|.$$

7.

Round	Proposals	Rejections	Strings
1	(α, C), (β, D), (γ, D), (δ, C), (ε, A)	(C, δ), (D, γ)	(A, ε), (C, α), (D, β)
2	(γ, A), (δ, B)	(A, ε)	(A, γ), (B, δ), (C, α), (D, β)
3	(ε, B)	(B, δ)	(A, γ), (B, ε), (C, α), (D, β)
4	(δ, D)	(D, β)	(A, γ), (B, ε), (C, α), (D, δ)
5	(β, C)	(C, α)	(A, γ), (B, ε), (C, β), (D, δ)
6	(α, A)	(A, α)	No change
7	(α, B)	(B, α)	No change
8	(α, D)	(D, δ)	(A, γ), (B, ε), (C, β), (D, α)
9	(δ, E)	None	(A, γ), (B, ε), (C, β), (D, α), (E, δ)

The final state of the strings after Step 9 is a stable matching.

8. Theorem 7.4 in the lecture notes.

9. (a) Definition 19.4 in the lecture notes.

(b) Theorem 19.8 in the lecture notes.