Tentamen
MVE505 Diskret Matematik TM1

2024-08-28 kl. 14.00–18.00

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Hjälpmedel: Inga

För godkänt på tentan krävs 45 poäng, inklusive eventuella bonuspoäng erhållna från inlämningsuppgifterna under VT-2024. Preliminärt så krävs 65 poäng för betyget 4 och 85 poäng för betyget 5. Dessa gränser kan minskas men inte höjas i efterhand.

Lösningar läggs ut på kursens Canvassida direkt efter tentan. Tentan rättas och bedöms anonymt. Resultatet meddelas i Ladok senast den 20 september. Granskning ordnas därefter av kursansvarig.

OBS! Alla stegen i dina resonemang måste motiveras väl i skrift och alla beräkningar visas. Det är i huvudsak tillvägagångssätten och motiveringarna som ger poäng, inte svaren.

I de uppgifter som består av fler olika delar går det alltid att lösa de enskilda delarna oberoende av varandra, även om man kan ibland spara räknetid genom att lösa deluppgifterna sekventiellt.

Om du i en lösning av någon av uppgifterna 1-7 åberopar en sats från kurslitteraturen så behöver du *inte* inkludera ett bevis av satsen.

I Uppgift 1 behöver du *inte* ange svaren som explicita decimaltal.

Jag bifogar 3 exemplar av Figure 6 så att ni har extra kladdpapper.

Var god vänd!

Uppgifterna

1. At the recent Olympics there were a total of 328 events.

(a) Suppose I told you the names of the 91 countries that each won at least one medal. How many possibilities does this leave for the list of countries winning *gold* medals ? (i.e.: we care about which country won gold in which event). (3p)

(b) Suppose instead I told you the names of the 91 countries that each won at least one medal, plus I told you that 63 of them won at least one gold, but I didn't tell you which 63. How many possibilities does this leave for the list which gives the number of gold medals per country ? (i.e.: we only care about *how many* golds each country won, not in which events). (3p)

(c) Sweden won 4 golds, 4 silvers and 3 bronzes, and did not win more than one medal in any event. How many possibilities does this leave for the choice of events in which Sweden won medals, assuming we care about which *value* of medal was won in each event ? (3p)

(d) Same question as in **(b)**, but this time I tell you which 63 countries won gold and, moreover, I tell you that no country won more than 150 golds (!!). (3p)

(e) There were 5 relay events in athletics[1]. Suppose these 5 golds were independently and randomly given to one of USA, UK and Jamaica. What is the probability that USA would end up with *strictly more* relay golds than UK ? (3p)

2. Compute an explicit formula for the numbers $(a_n)_{n=0}^{\infty}$ which satisfy the recursion (12p)

$$a_0 = 1, \;\; a_1 = 1, \quad a_n = 9a_{n-1} - 8a_{n-2} + 3 \cdot 2^n + 14n, \;\; \forall \, n \geq 2.$$

3. Compute the general solution to the Diophantine equation (8p)

$$353x + 77y = 100,$$

along with all solutions satisfying $|x| + |y| < 1000$.

4. (a) Without doing *any* calculations, explain why at least one of the nmbers 16439, 16441 and 16443 cannot be prime. (3p)

(b) Turns out 16443 ain't prime. Determine the number of elements in the ring $\mathbb{Z}_{16443}$ that have a multiplicative inverse. (5p)

5. Compute the general solution, plus the greatest negative solution, to the following system of congruences: (8p)

$$3x \equiv 1 \pmod{7}, \quad 5x \equiv 2 \pmod{8}, \quad 6x \equiv 5 \pmod{13}.$$

[1] 4×100 for men and women, 4×400 for men and women, plus a 4×400 mixed relay.

6. Let G be the network in Figure 6, let G^* be the underlying undirected, but still weighted graph and let G^{**} be the underlying undirected and unweighted graph.

(a) Determine, with proof, $\chi(G^{**})$ (Obs! You can appeal to theorems in the lecture notes without proof). (3p)

(b) Add in as few edges to G^{**} as possible so that the resulting graph or multigraph (you are allowed to create multiple edges !) has an Euler trail. Then determine an explicit Euler trail in the resulting (multi)graph. (4p)

(c) Starting from s, apply Prim's algorithm to determine a minimal spanning tree in G^*. Indicate clearly which edge is chosen at each step and draw the final tree. (4p)

(d) Implement the Ford-Fulkerson algorithm to determine a maximum flow from s to t in the network G and a corresponding minumum cut. Write clearly which f-augmenting path you choose at each step. Draw the final flow in full and indicate the corresponding minimum cut. (7p)

7. Let G be a simple graph on $n \geq 1$ nodes and suppose that $\deg(v) > \frac{n-2}{2}$ for each $v \in V(G)$. Prove that G must be connected. (7p)

8. (a) Define the Catalan numbers C_n, where n is a non-negative integer. (2p)

(b) State and prove an explicit formula, in terms of binomial coefficients, for these numbers. (Obs! It's up to you *how* you prove the formula.) (10p)

9. (a) Define clearly what is meant by a *stable matching* for a *bipartite dataset* (Obs! What you write should clearly explain *both* terms). (2p)

(b) Describe the Gale-Shapley algorithm, and prove that it always produces a stable matching for a bipartite dataset. (10p)

Go n'eirí an bóthar libh!

Lösningar Diskret Matematik TM1, 240828

1. (a) We have 91 choices for each gold medal and 328 gold medals, thus 91^{328} possibilities.

   (b) First there are $\binom{91}{63}$ choices for the countries that won gold(s). Having chosen these, first give each country one gold. We then have $328 - 63 = 265$ golds which can be freely distributed amongst the 63 countries. This can be done in $\binom{265+63-1}{63-1} = \binom{327}{62}$ ways.

   Hence, by MP, the total number of possibilities is $\binom{91}{63} \times \binom{327}{62}$.

   (c) This is equivalent to the number of possibilities for a 328-letter word consisting of 4 G's, 4 S's, 3 B's and 317 N's (N = nothing). By Theorem 2.6, the number of such words is $\frac{328!}{4!\,4!\,3!\,317!}$.

   (d) If I tell you which 63 countries won gold then, by the same reasoning as in **(b)**, this leaves $\binom{327}{62}$ possibilities, if no further restrictions are placed. Let Ω be this set of options. We seek $|\Omega \backslash X|$, where X is the set of options where some country wins 151 or more golds. List the 63 countries in any order and let X_i be the set of options in which country i wins at least 151 golds. Note that the intersection of any two X_i will be empty, since $151 \cdot 2 > 265$. Thus, $|X| = \sum |X_i| = 63|X_1|$. To compute $|X_1|$, first give 151 golds to country nr. 1 and 1 gold to every other country. We then have $328 - 151 - 62 = 115$ golds which can be distributed freely amongst the 63 countries,and thus $|X_1| = \binom{115+63-1}{63-1} = \binom{177}{62}$.

   Answer: $\binom{327}{62} - 63 \cdot \binom{177}{62}$.

   (e) Let Ω be the set of all possible outcomes. $|\Omega| = 3^5$, since there are 3 options for each of the 5 golds. Let A be the event that USA wins more golds than UK, let B be the reverse event and let C be the event that they win the same number of golds. Then A, B and C are mutually exclusive and together cover Ω. We seek $\mathbb{P}(A)$ and, by symmetry, $\mathbb{P}(A) = \mathbb{P}(B)$. Thus,

   $$\mathbb{P}(A) = \frac{1}{2}(1 - \mathbb{P}(C)). \tag{1}$$

   Further, we have $\mathbb{P}(C) = N/3^5$, where N is the number of outcomes in which USA and UK win the same number of golds. This number can be 0, 1 or 2 and hence

   $$N = N_0 + N_1 + N_2 = 1 + \binom{2}{1}\binom{5}{3} + \binom{4}{2}\binom{5}{1} = 1 + 20 + 30 = 51.$$

   Substituting into (1), we get

   $$\mathbb{P}(A) = \frac{1}{2}\left(1 - \frac{51}{243}\right) = \frac{32}{81}.$$

2. *Step 1:* The characteristic equation is $x^2 = 9x - 8$, which has the roots $x_1 = 1$, $x_2 = 8$. Hence the general solution of the homogeneous equation is

   $$a_{h,n} = C_1 + C_2 \cdot 8^n.$$

   Step 2: Since 1 is a root of multiplicity one of the characteristic equation, whereas 2 is not a root at all, our choice of a particular solution should look like

   $$a_{p,n} = a_{p_1,n} + a_{p_2,n},$$

where

$$a_{p_1,n} = C_3 \cdot 2^n, \quad a_{p_2,n} = n(C_4 n + C_5) = C_4 n^2 + C_5 n.$$

Inserting into the recurrence gives, firstly,

$$C_3 \cdot 2^n = 9C_3 \cdot 2^{n-1} - 8C_3 \cdot 2^{n-2} + 3 \cdot 2^n.$$

Cancelling 2^{n-2} gives

$$4C_3 = 18C_3 - 8C_3 + 12 \Rightarrow C_3 = -2.$$

Secondly,

$$C_4 n^2 + C_5 n = 9[C_4(n-1)^2 + C_5(n-1)] - 8[C_4(n-2)^2 + C_5(n-2)] + 14n.$$

The coefficients of n^2 will cancel exactly. Equating coefficients of n yields

$$C_5 = 9(-2C_4 + C_5) - 8(-4C_4 + C_5) + 14 \Rightarrow \cdots \Rightarrow C_4 = -1.$$

Finally, equating constant coefficients yields

$$0 = 9(C_4 - C_5) - 8(4C_4 - 2C_5) \stackrel{C_4=-1}{\Rightarrow} C_5 = -\frac{23}{7}.$$

Step 3: Hence, the general solution of the recurrence is

$$a_n = a_{h,n} + a_{p_1,n} + a_{p_2,n} = C_1 + C_2 \cdot 8^n - 2^{n+1} - n^2 - \frac{23n}{7}.$$

Insert the initial conditions:

$$n = 0: \quad a_0 = 1 = C_1 + C_2 - 2 \Rightarrow C_1 + C_2 = 3,$$
$$n = 1: \quad a_1 = 1 = C_1 + 8C_2 - 4 - 1 - \frac{23}{7} \Rightarrow C_1 + 8C_2 = \frac{65}{7}.$$

We solve easily to get $C_1 = 103/49$, $C_2 = 44/49$. Hence,

$$a_n = \frac{103}{49} + \frac{44}{49} \cdot 8^n - 2^{n+1} - n^2 - \frac{23n}{7}.$$

3. First run Euclid's algorithm forwards to find GCD(353, 77):

$$\begin{aligned} 353 &= 4 \cdot 77 + 45, \\ 77 &= 1 \cdot 45 + 32, \\ 45 &= 1 \cdot 32 + 13, \\ 32 &= 2 \cdot 13 + 6, \\ 13 &= 2 \cdot 6 + 1. \end{aligned}$$

So GCD(353, 77) $= 1$, which means the Diophantine equation has a solution, since the right-hand side is a multiple of 1. The general solution has the form

$$x = mx_0 - \left(\frac{b}{d}\right) n, \quad y = my_0 + \left(\frac{a}{d}\right) n, \quad n \in \mathbb{Z}. \tag{2}$$

Here $a = 353$, $b = 77$, $d = 1$ and $m = c/d = 100$. To find (x_0, y_0), we run backwards through Euclid:

$$\begin{aligned} 1 &= 13 - 2 \cdot 6 \\ &= 13 - 2(32 - 2 \cdot 13) \\ &= 5 \cdot 13 - 2 \cdot 32 \\ &= 5(45 - 32) - 2 \cdot 32 \\ &= 5 \cdot 45 - 7 \cdot 32 \\ &= 5 \cdot 45 - 7(77 - 45) \\ &= -7 \cdot 77 + 12 \cdot 45 \\ &= -7 \cdot 77 + 12(353 - 4 \cdot 77) \\ \Rightarrow 1 &= 12 \cdot 353 - 55 \cdot 77. \end{aligned}$$

Thus $x_0 = 12$, $y_0 = -55$. Inserting into (2) gives

$$x = 1200 - 77n, \quad y = -5500 + 353n, \quad n \in \mathbb{Z}.$$

It is then easy to check that there are 4 solutions satisfying $|x| + |y| < 1000$. These correspond to $n = 14, 15, 16, 17$ and are, respectively,

$$(122, -558), \quad (45, -205), \quad (-32, 148), \quad (-109, 501).$$

4. (a) Any three consecutive odd numbers are mutually incongruent modulo 3, hence one of them must be a multiple of 3.

(b) We seek $\phi(16443)$. First we factorize. Using the digit-sum trick, we get

$$16443 = 3 \cdot 5481 = 3^2 \cdot 1827 = 3^3 \cdot 609 = 3^4 \cdot 203 = 3^4 \cdot 7 \cdot 29.$$

Hence,

$$\phi(16443) = \phi(3^4) \cdot \phi(7) \cdot \phi(29) = (3^4 - 3^3)(7 - 1)(29 - 1) = 54 \cdot 6 \cdot 28 = 9072.$$

5. First some editing:

$$\begin{aligned} 3x \equiv 1 \pmod 7 &\Rightarrow x \equiv 3^{-1} \cdot 1 \equiv (-2) \cdot 1 \equiv -2 \pmod 7, \\ 5x \equiv 2 \pmod 8 &\Rightarrow x \equiv 5^{-1} \cdot 2 \equiv 5 \cdot 2 \equiv 10 \equiv 2 \pmod 8, \\ 6x \equiv 5 \pmod{13} &\Rightarrow x \equiv 6^{-1} \cdot 5 \equiv (-2) \cdot 5 \equiv -10 \equiv 3 \pmod{13}. \end{aligned}$$

Thus, by eq. (11.3) in the lecture notes, the general solution is

$$x \equiv -2 \cdot b_1 \cdot 8 \cdot 13 + 2 \cdot b_2 \cdot 7 \cdot 13 + 3 \cdot b_3 \cdot 7 \cdot 8 \pmod{7 \cdot 8 \cdot 13}, \tag{3}$$

where

$$\begin{aligned} b_1 &\equiv (8 \cdot 13)^{-1} \equiv (1 \cdot (-1))^{-1} \equiv (-1)^{-1} \equiv -1 \pmod 7, \\ b_2 &\equiv (7 \cdot 13)^{-1} \equiv ((-1) \cdot (-3))^{-1} \equiv 3^{-1} \equiv 3 \pmod 8, \\ b_3 &\equiv (7 \cdot 8)^{-1} \equiv 56^{-1} \equiv 4^{-1} \equiv -3 \pmod{13}. \end{aligned}$$

We thus choose $b_1 = -1$, $b_2 = 3$, $b_3 = -3$ and insert into (3) to get

$$\begin{aligned} x &\equiv (-2)\cdot(-1)\cdot 8\cdot 13 + 2\cdot 3\cdot 7\cdot 13 + 3\cdot(-3)\cdot 7\cdot 8 \\ &\equiv 208 + 546 - 504 \equiv 250 \pmod{728}. \end{aligned}$$

So the general solution is $x \equiv 250 \pmod{728}$. We do a sanity check:

$250 - (-2) = 252 = 7 \cdot 36$, ok

$250 - 2 = 248 = 8 \cdot 31$, ok

$250 - 3 = 247 = 13 \cdot 19$, ok !
Finally, the greatest negative solution is $x = 250 - 728 = -478$.

6. (a) Perhaps the most natural thing to try first is a greedy coloring, starting with s, then in alphabetical order, and ending with t. If so, we'll use color 2 at a and need color 4 at f. But if we switch to color 3 at a, then we can use color 2 at f and continue using only 3 colors in all:

Color 1: s, d, g, h, t.

Color 2: b, f, i.

Color 3: a, c, e, j.

Hence $\chi(G^{**}) \leq 3$. But G^{**} contains many triangles, hence $\chi(G^{**}) = 3$.

(b) There are 6 nodes of odd degree: s, a, b, e, g, t. I thus need to add 2 edges in order to reduce this number to two. For example, let's add another $\{s, b\}$ edge and another $\{e, g\}$ edge. Then only a and t have odd degree, so there must be an Euler trail between them. An example of such a trail is as follows:

$$\begin{aligned} & a \to s \to b \to s \to c \to b \to d \to a \to f \to d \to e \to f \to t \\ & \to i \to e \to b \to g \to c \to h \to j \to g \to e \to g \to i \to j \to t. \end{aligned}$$

(c) The algorithm could proceed as follows (there are other possibilities, for example the edge $\{a, f\}$ could be chosen at Step 2):

Step	Edge chosen	Weight
1	$\{s, a\}$	9
2	$\{a, d\}$	4
3	$\{d, b\}$	3
4	$\{d, e\}$	3
5	$\{e, g\}$	3
6	$\{b, c\}$	4
7	$\{e, f\}$	4
8	$\{e, i\}$	4
9	$\{i, j\}$	1
10	$\{c, h\}$	5
11	$\{j, t\}$	8
Total weight		48

The final tree is illustrated in Figure L.6(c).

Step	f-augmenting path	Increase in flow strength
1	$s \to a \to f \to t$	4
2	$s \to a \to d \to f \to t$	4
3	$s \to b \to g \to i \to t$	8
4	$s \to c \to h \to j \to t$	5
5	$s \to c \to g \to j \to t$	3
6	$s \to c \to g \to j \to i \to t$	1
7	$s \to b \to e \to f \to t$	2
Total flow strength		27

(d) The algorithm could proceed as above (there are other alternatives).
The final flow is illustrated in Figure L.6(d). The set of nodes that can now be reached from s via an augmenting path is $S = \{s, a\}$. Sätt $T = V \setminus S = \{b, c, d, e, f, g, h, i, j\, t\}$. Then we have

$$c(S, T) = c(a, d) + c(a, f) + c(s, b) + c(s, c) = 4 + 4 + 10 + 9 = 27 = |f|.$$

7. Let x and y be two nodes in G. It suffices to show there must be a path in G between x and y. If there is an edge between them, we are done. Otherwise, there are $n-2$ nodes in G, other than x and y, and every neighbor of either one is amongst those $n-2$ nodes. But since each has degree greater than $\frac{n-2}{2}$, it must be the case (by the Pigeonhole Principle) that they have at least one common neighbor, say z. Then there is a path of length 2 between them, via z.

8. (a) Definitions 5.7, 5.8 and 5.9 in the lecture notes.

(b) Theorem 5.10 in the lecture notes. Either of the two proofs given in the notes is acceptable.

9. (a) Dataset 21.2 and Definition 21.2 in the lecture notes.

(b) Theorem 21.3 in the lecture notes.