Tentamen
MVE505 Diskret Matematik TM1

2023-10-06 kl. 14.00–18.00

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Hjälpmedel: Inga

För godkänt på tentan krävs 45 poäng, inklusive eventuella bonuspoäng erhållna från inlämningsuppgifterna under VT-2023. Preliminärt så krävs 65 poäng för betyget 4 och 85 poäng för betyget 5. Dessa gränser kan minskas men inte höjas i efterhand.

Lösningar läggs ut på kursens Canvassida direkt efter tentan. Tentan rättas och bedöms anonymt. Resultatet meddelas i Ladok senast den 27 oktober. Granskning ordnas därefter av kursansvrig.

OBS! Alla stegen i dina resonemang måste motiveras väl i skrift och alla beräkningar visas. Det är i huvudsak tillvägagångssätten och motiveringarna som ger poäng, inte svaren.

I de uppgifter som består av fler olika delar går det alltid att lösa de enskilda delarna oberoende av varandra, även om man kan ibland spara räknetid genom att lösa deluppgifterna sekventiellt.

Om du i en lösning av någon av uppgifterna 1-6 åberopar en sats från kurslitteraturen så behöver du *inte* inkludera ett bevis av satsen.

I Uppgift 1 behöver du *inte* ange svaren som explicita decimaltal.

Jag bifogar 3 exemplar av Figur 5B så att ni har extra kladdpapper.

Var god vänd!

Uppgifterna

1. Sju olika familjer A-G har i sina hem sammanlagt 15 TV-apparater.

(a) Hur många möjligheter finns det för antalet TV-apparater per hus om

i. Vi vet att varje familj äger minst en apparat ? (3p)

ii. Vi vet att familj A äger högst 2 apparat, och att exakt 2 av de övriga familjerna (vi vet inte vilka) äger ingen apparat alls ? (3p)

(b) Om vi vet ingenting om fördelningen av apparaterna, vad är från början sannolikheten att familjerna A, B, C och D alla äger samma antal apparater ? (3p)

(c) I stans butik finns fyra olika märken av TV-apparater och för varje märke finns både en smart-TV och en dum-TV. Om vi ska ha hänsyn till vilka sorts TV-apparater varje familj äger, hur många möjligheter finns det totalt om

i. varje familj äger minst 2 apparater ? (3p)

ii. varje familj äger minst 2 apparater, apparaterna i varje hus är alla av samma märke och inget märke ägs av fler än 2 familjer ? (3p)

2. Bestäm en explicit formel för talen $(a_n)_{n=0}^{\infty}$ som uppfyller rekursionen (12p)

$$a_0 = a_1 = a_2 = 1, \quad a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} + 1, \ \forall\, n \geq 3.$$

(Tips: $1 + 8 = 5 + 4$).

3. (a) För vilka $c \in \mathbb{Z}$ har den Diofantiska ekvationen (9p)

$$369x + 2598y = c$$

en lösning ?

(b) Bestäm för $c = 30$ en formel för den allmänna lösningen, samt den lösning som minimerar $|x| + |y|$.

4. Bestäm den allmäna lösningen samt den minsta positiva lösningen till följande system av kongruenser: (9p)

$$3x \equiv 1 \pmod 8, \quad 5x \equiv 2 \pmod{11}, \quad 7x \equiv 3 \pmod{17}.$$

Var god vänd!

5. (a) För varje par av grafer i Figur 5A, avgör om graferna är isomorfa eller ej. Om de inte är det, säg varför, och om de är det ange en explicit isomorfi dem emellan. (5p)

(b) Låt G vara nätverket i Figur 5B och låt G^* vara den underliggande oriktade och oviktade grafen.

i. Bestäm med bevis det kromatiska talet hos G^* samt ange en explicit optimal färgläggning. (4p)

ii. Lägg till så få kanter som möjligt till G^* så att den resulterande grafen är fortsatt enkel men har en Eulerväg. Ange sedan en sådan väg i den nya grafen. (4p)

iii. Implementera Ford-Fulkerson algoritmen för att bestämma ett maximalt flöde från s till t i nätverket G och en motsvarande minumum cut. Skriv tydligt vilken f-augmenterande stig du väljer i varje steg. Rita det slutgiltiga flödet i sin helhet och indikera motsvarande minimum cut. (7p)

6. Bevisa att, för alla $n \geq 3$ gäller (10p)

$$S(n, 3) = \frac{1}{2}(3^{n-1} + 1) - 2^{n-1},$$

där $S(\cdot, \cdot)$ betecknar ett Stirlingtal av andra arten.

7. (a) Definiera Eulers phi-funktion $\phi : \mathbb{N} \to \mathbb{N}$. (1.5p)

(b) Formulera Eulers sats. (1.5p)

(c) Bevisa Eulers sats. (10p)

8. (a) Formulera *tydligt* hur Prims algoritm för att hitta ett MST i en viktad, sammanhängande graf går till. (2p)

(b) Bevisa att algoritmen alltid producerar ett MST. (10p)

Go n'eirí an bóthar libh!

Lösningar Diskret Matematik TM1, 231006

1. (a) i. We are to place 15 indistinguishable balls (the TVs) into 7 distinguishable bins (the families). Since each bin must have at least one ball, we first place one ball in each bin and then have 8 balls left which we can distribute freely. By Proposition 1.9, the number of ways this can be done is $\binom{8+7-1}{7-1} = \binom{14}{6}$.

ii. First of all, there are $\binom{6}{2}$ possibilities for the other two families without any TVs. Having chose those, we now have to distribute 15 TVs amongst five families, including A. For each $i = 0, 1, 2$, if family A gets i TVs, we'll be left with $15 - i$ TVs for the other 4 families. But each remaining family must get at least one TV, so we'll only have $11 - i$ TVs to distribute freely amongst these four families. Hence, the total number of possibilities for distributing the TVs is

$$\binom{6}{2} \cdot \sum_{i=0}^{2} \binom{(11-i)+4-1}{4-1} = \binom{6}{2} \cdot \left[\binom{14}{3} + \binom{13}{3} + \binom{12}{3} \right].$$

(b) The desired probability is $N/\binom{14}{6}$, where N is the number of ways to distribute the TVs so that families A, B, C and D all get the same number. Since there are 15 TVs, this number must be either 0, 1, 2 or 3. If this number is i, then we'll have $15 - 4i$ TVs to divide amongst the remaining three families E, F and G. The number of ways this can be done is $\binom{(15-4i)+3-1}{3-1} = \binom{17-4i}{2}$. Hence, the desired probability is

$$\frac{1}{\binom{14}{6}} \sum_{i=0}^{3} \binom{17-4i}{2} = \frac{\binom{17}{2} + \binom{13}{2} + \binom{9}{2} + \binom{5}{2}}{\binom{14}{6}}.$$

(c) i. Since each family owns at least 2 TVs, there must be exactly one family that owns 3 of them and every other family owns two. There are 7 choices for which family has an extra TV. Once that family is chosen, we simply have to choose the brand and "intelligence" of each TV. There are $4 \times 2 = 8$ choices per TV, hence a total of $7 \cdot 8^{15}$ possibilities.

ii. There are still 7 choices for the family with the extra TV. Next, we have to assign a brand to each family. Since no brand is assigned more than twice, it means that three brands must be assigned twice and one brand assigned once. There are 4 choices for the "unpopular" brand and, once this is chosen, there are $\frac{7!}{2!\,2!\,2!\,1!} = \frac{7!}{8}$ choices for how to distribute the brands amongst the families.
Finally, once the numbers and brands of TVs per family are chosen, we have two choices for the "intelligence" of each TV, giving 2^{15} possible collections of TVs. Putting it all together, and using MP, the total number of possibilities is

$$7 \cdot 4 \cdot \frac{7!}{8} \cdot 2^{15}.$$

2. *Step 1:* The characteristic equation is $x^3 = 5x^2 - 8x + 4$. Using the hint, we see that $x_1 = 1$ is a root. Polynomial disivion gives

$$x^3 - 5x^2 + 8x - 4 = (x-1)(x^2 - 4x + 4) = (x-1)(x-2)^2,$$

so the other repeated root is $x_{2,3} = 2$. Hence the general solution of the homogeneous equation is

$$a_{h,n} = C_1 + (C_2 + C_3 n)2^n.$$

Step 2: Since 1 is a root of multiplicity one of the characteristic equation, our choice of a particular solution should look like

$$a_{p,n} = C_4 n.$$

Inserting into the recurrence gives

$$C_4 n = 5C_4(n-1) - 8C_4(n-2) + 4C_4(n-3) + 1.$$

The coefficients of n will cancel exactly. Comparing constant coefficients yields $C_4 = 1$.

Step 3: Hence, the general solution of the recurrence is

$$a_n = a_{h,n} + a_{p,n} = C_1 + (C_2 + C_3 n)2^n + n.$$

Insert the initial conditions:

$$\begin{aligned} n = 0: \quad a_0 = 1 = C_1 + C_2 &\Rightarrow C_1 + C_2 = 1, \\ n = 1: \quad a_1 = 1 = C_1 + 2(C_2 + C_3) + 1 &\Rightarrow C_1 + 2C_2 + 2C_3 = 0, \\ n = 2: \quad a_2 = 1 = C_1 + 4(C_2 + 2C_3) + 2 &\Rightarrow C_1 + 4C_2 + 8C_3 = -1. \end{aligned}$$

We solve easily to get $C_1 = 3$, $C_2 = -2$, $C_3 = 1/2$. Hence,

$$a_n = 3 + n + (n-4)2^{n-1}.$$

3. (a) First run Euclid's algorithm forwards to find GCD(2598, 369):

$$\begin{aligned} 2598 &= 7 \cdot 369 + 15, \\ 369 &= 24 \cdot 15 + 9, \\ 15 &= 1 \cdot 9 + 6, \\ 9 &= 1 \cdot 6 + 3, \\ 6 &= 2 \cdot 3 + 0. \end{aligned}$$

So GCD(2598, 369) = 3, which means the Diophantine equation has a solution if and only if c is a multiple of 3.

(b) First run Euclid backwards:

$$\begin{aligned} 3 &= 9 - 6 \\ &= 9 - (15 - 9) \\ &= 2 \cdot 9 - 15 \\ &= 2(369 - 24 \cdot 15) - 15 \\ &= 2 \cdot 369 - 49 \cdot 15 \\ &= 2 \cdot 369 - 49(2598 - 7 \cdot 369) \end{aligned}$$

$$\Rightarrow 3 = -49 \cdot 2598 + 345 \cdot 369.$$

So take $x_0 = -49$, $y_0 = 345$. The general solution of the Diophantine equation has the form

$$x = mx_0 + \left(\frac{b}{d}\right) n, \qquad y = my_0 - \left(\frac{a}{d}\right) n, \quad n \in \mathbb{Z}.$$

We have $a = 2598$, $b = 369$ and $d = 3$. For $c = 30$, we have $m = c/d = 10$. Inserting everything gives

$$x = -490 + 123n, \qquad y = 3450 - 866n, \quad n \in \mathbb{Z}.$$

If we take $n = 4$, we get the solution $x = 2$, $y = -14$ and this is clearly the solution which minimizes $|x| + |y|$.

4. First some editing:

$$\begin{aligned} 3x \equiv 1 \pmod 8 &\Rightarrow x \equiv 3^{-1} \cdot 1 \equiv 3 \cdot 1 \equiv 3 \pmod 8, \\ 5x \equiv 2 \pmod{11} &\Rightarrow x \equiv 5^{-1} \cdot 2 \equiv (-2) \cdot 2 \equiv -4 \pmod{11}, \\ 7x \equiv 3 \pmod{17} &\Rightarrow x \equiv 7^{-1} \cdot 3 \equiv 5 \cdot 3 \equiv 15 \equiv -2 \pmod{17}. \end{aligned}$$

Thus, by eq. (11.3) in the lecture notes, the general solution is

$$x \equiv 3 \cdot b_1 \cdot 11 \cdot 17 - 4 \cdot b_2 \cdot 8 \cdot 17 - 2 \cdot b_3 \cdot 8 \cdot 11 \pmod{8 \cdot 11 \cdot 17}, \qquad (1)$$

where

$$\begin{aligned} b_1 &\equiv (11 \cdot 17)^{-1} \equiv (3 \cdot 1)^{-1} \equiv 3^{-1} \equiv 3 \pmod 8, \\ b_2 &\equiv (8 \cdot 17)^{-1} \equiv (8 \cdot 6)^{-1} \equiv 48^{-1} \equiv 4^{-1} \equiv 3 \pmod{11}, \\ b_3 &\equiv (8 \cdot 11)^{-1} \equiv 88^{-1} \equiv 3^{-1} \equiv 6 \pmod{17}. \end{aligned}$$

We thus choose $b_1 = 3$, $b_2 = 3$, $b_3 = 6$ and insert into (1) to get

$$\begin{aligned} x &\equiv 3 \cdot 3 \cdot 11 \cdot 17 - 4 \cdot 3 \cdot 8 \cdot 17 - 2 \cdot 6 \cdot 8 \cdot 11 \\ &\equiv 1683 - 1632 - 1056 \equiv -1005 \equiv 491 \pmod{1496}. \end{aligned}$$

ANSWER: $x \equiv 491 \pmod{1496}$.
SANITY CHECK: Check that $x = 491$ satisfies the original three congruences by direct calculation:

$491 - 3 = 488 = 8 \cdot 61$, ok

$491 - (-4) = 495 = 11 \cdot 45$, ok

$491 - (-2) = 493 = 17 \cdot 29$, ok !

5. (a) The first pair are isomorphic, for example with the vertices labelled as in Figure L.5A. The seond pair are not since, for example, the graph on the left has two distinct K_3:s, whereas the one on the right has only one.

(b) i. Since the graph is plane, we must have $\chi(G) \leq 4$. On the other hand, it contains a wheel subgraph W_5 centered at b and with spokes out to s, a, d, e, c. This subgraph already requires 4 colors and so $\chi(G) = 4$. An explicit 4-coloring is got, for example, by coloring greedily starting at s, then in alphabetical order. This will yield the following coloring:

$$\text{Color } 1: \ s, e, f. \quad \text{Color } 2: \ a, c, g. \quad \text{Color } 3: \ b, h. \quad \text{Color } 4: \ d, t.$$

ii. Only s and d have even degree. Since we have 8 nodes of odd degree, and need to have only 2 such nodes for there to exist an Euler trail, we must add a minimum of 3 edges. This is doable, for example if we add the edges $\{a, f\}$, $\{c, h\}$ and $\{e, t\}$, then only b and g will have odd degree. An example of an Euler trail between these two nodes will then be

$$b \to s \to d \to a \to s \to c \to b \to a \to f \to t \to g \to f$$
$$\to d \to b \to e \to d \to g \to e \to c \to h \to e \to t \to h \to g.$$

iii. The algorithm could proceed as follows (there are other alternatives).

Step	f-augmenting path	Increase in flow strength
1	$s \to d \to f \to t$	5
2	$s \to a \to d \to g \to t$	4
3	$s \to b \to d \to g \to t$	2
4	$s \to b \to e \to h \to t$	5
5	$s \to c \to e \to h \to t$	1
6	$s \to c \to e \to d \to f \to t$	2
7	$s \to c \to e \to g \to h \to t$	1
Total flow strength		21

The final flow is illustrated in Figure L.5B. All arcs to t are saturated and every node except t can be reached from s via an augmenting path. Note that, in order to reach the triangle $\{a, b, d\}$, we have to remove flows along negatively oriented arcs. Set $T = \{t\}$, $S = V \backslash \{t\}$. Then we have

$$c(S, T) = c(f, t) + c(g, t) + c(h, t) = 7 + 6 + 8 = 21 = |f|.$$

6. $S(n, 3)$ is the number of ways to distribute n distinguishable balls into 3 indistinguishable bins so that no bin is left empty.

First, let us assume the bins are instead distinguishable and consider the number of ways to distribute the balls. There are 3 choices for each ball, thus a total of 3^n ways to distribute them, without preconditions. If no bin is to be left empty, then we disallow the following distributions:

(i) those in which all the balls are put in one bin. There are 3 such distributions.

(ii) those in which exactly one bin is left empty. There are 3 choices for the empty bin. Once this is chosen, we have 2 choices left for each ball, giving a total of 2^n distributions. But here agin, we must disallow the 2 distributions where all n balls end up in the same bin. Thus, we disallow a total of $3(2^n - 2)$ distributions in this case.

Hence, the total number of distributions of distinguishable balls into *distinguishable* bins so that no bin is left empty is

$$3^n - 3 - 3(2^n - 2) = 3 \cdot (3^{n-1} + 1 - 2 \cdot 2^{n-1}).$$

If we now instead consider indistinguishable bins, it means that any of the $3! = 6$ permutations of the *distinguishable* bins doesn't change the distribution of balls. Hence,

$$S(n,\, 3) = \frac{3 \cdot (3^{n-1} + 1 - 2 \cdot 2^{n-1})}{6} = \frac{3^{n-1} + 1}{2} - 2^{n-1}, \quad \text{v.s.v.}$$

7. Definition 11.6 and Theorem 11.7 in the lecture notes.

8. Theorem 18.8 in the lecture notes. The algorithm is described in the text before the theorem.