Tentamen
MVE505 Diskret Matematik TM1

2023-08-23 kl. 14.00–18.00

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Hjälpmedel: Inga

För godkänt på tentan krävs 45 poäng, inklusive eventuella bonuspoäng erhållna från inlämningsuppgifterna under VT-2023. Preliminärt så krävs 65 poäng för betyget 4 och 85 poäng för betyget 5. Dessa gränser kan minskas men inte höjas i efterhand.

Lösningar läggs ut på kursens Canvassida direkt efter tentan. Tentan rättas och bedöms anonymt. Resultatet meddelas i Ladok senast den 14 september. Granskning ordnas därefter av kursansvrig.

OBS! Alla stegen i dina resonemang måste motiveras väl i skrift och alla beräkningar visas. Det är i huvudsak tillvägagångssätten och motiveringarna som ger poäng, inte svaren.

I de uppgifter som består av fler olika delar går det alltid att lösa de enskilda delarna oberoende av varandra, även om man kan ibland spara räknetid genom att lösa deluppgifterna sekventiellt.

Om du i en lösning av någon av uppgifterna 1-7 åberopar en sats från kurslitteraturen så behöver du *inte* inkludera ett bevis av satsen.

I Uppgift 1 behöver du ange svaren som explicita decimaltal *endast i* deluppgift (b).

Jag bifogar 3 exemplar av Figur 6 så att ni har extra kladdpapper.

Var god vänd!

Uppgifterna

1. An American, a Russian and a Swede walk into a bar. The bar has 10 different brands of each of beer, wine and vodka. They order 4 rounds of drinks before heading off merrily to watch some cage fighting.

(a) First of all, suppose we care about *who* orders *which* brands, but not the order of the rounds. How many possibilities does this leave for the brands of the 12 drinks ordered by the party

i. if nobody ordered the same brand more than once (but different people may have ordered the same brand) ? (3p)

ii. in total (i.e.: without the previous restriction, but where we still take account of *how many* drinks of each brand each person ordered) ? (3p)

iii. if nobody ordered the same brand more than once and, in addition, we know that (3p)
- the American had a Bud and a Coors
- the Russian had *at least* two vodkas
- the Swede is a single cat-owner who only ever drinks wine.

(b) Suppose instead all we care about is the total number of beers, wines and vodkas consumed amongst the 12 drinks i.e.: it no longer matters who ordered what or exactly which brands were ordered. How many possibilities are there for this

i. in total ? (3p)

ii. if no brand of anything was ordered by more than one person ? (3p)

2. Determine an explicit formula for the numbers $(a_n)_{n=0}^{\infty}$ which satisfy the recursion (12p)

$$a_0 = 3, \quad a_1 = 8, \qquad a_n = 6a_{n-1} - 9a_{n-2} + 3^n + 4(n-1), \quad \forall\, n \geq 2.$$

3. Determine the general solution, as well as the largest negative solution, to the following system of congruences: (8p)

$$2x \equiv 3 \pmod 7, \qquad 3x \equiv 2 \pmod{11}, \qquad 8x \equiv 7 \pmod{13}.$$

4. (a) Compute $3^{158} \pmod{17}$

i. using Fermat (5p)

ii. by repeated squaring. (5p)

(b) Is 3 a primitive root modulo 17 ? Explain. (3p)

(Obs! In part (a), you are meant to show how the respective procedure works. You'll get *zero* points for computing some other way, even if the answer is correct).

Var god vänd!

5. Determine a 2-parameter formula for the general solution to the Diophantine equation (8p)

$$39x + 52y + 84z = 1.$$

6. Let G be the network in Figure 6 and let G^* be the underlying undirected graph.

(a) Use either Prim's or Kruskal's algorithm to determine a minimal spanning tree in G^*. Indicate *clearly* which edge is chosen at each step of the procedure and draw the final MST. (5p)

(b) Use Dijkstra's algorithm to find a shortest path from s to t in G^* (*still* the undirected graph !). Indicate *clearly* which edge is chosen and which label is set at each step of the procedure. Write down the obtained path. (6p)

(c) Implement the Ford-Fulkerson algorithm to determine a maximum flow from s to t in G and a corresponding minimum cut. Indicate *clearly* which f-augmenting path you choose at each step of the procedure. Draw the final flow in full and indicate the corresponding cut. (7p)

7. (a) Let n be a positive integer. Define the Catalan number C_n. (Obs! If your definition is in terms of *Dyck paths*, then you must also explain what those are). (2p)

(b) Prove that (11p)

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Obs! It's up to you whether to prove this using generating functions or some other method.

8. (a) Explain what is meant by a *stable matching* for a *bipartite dataset*. (2p)

(b) Describe the Gale-Shapley algorithm and prove that it always produces a stable matching for such a dataset. (11p)

Go n'eirí an bóthar libh!

Lösningar Diskret Matematik TM1, 230823

1. (a) i. Each person orders 4 brands out of 30, so has $\binom{30}{4}$ options. By MP, the number of options for the trio is $\binom{30}{4}^3$.

ii. Each person places 4 identical balls (the unordered drinks) into 30 distinguishable boxes (the brands). He thus has $\binom{30+4-1}{4} = \binom{33}{4}$ options. The number of options for the trio is thus $\binom{33}{4}^3$.

iii. The Swede has $\binom{10}{4}$ options for his 4 brands of wine. The American has to choose 2 other brands from among 28, so has $\binom{28}{2}$ options. If the Russian drinks $i \in \{2, 3, 4\}$ vodkas then he has $\binom{10}{i}\binom{20}{4-i}$ options. By MP and AP, the total number of options for the trio is

$$\binom{10}{4} \cdot \binom{28}{2} \cdot \left[\binom{10}{2}\binom{20}{2} + 20\binom{10}{3} + \binom{10}{4}\right].$$

(b) i. We are placing 12 identical balls into the 3 distinguishable boxes "beer", "wine" and "vodka". Hence there are $\binom{12+3-1}{3-1} = \binom{14}{2} = \frac{14 \cdot 13}{1 \cdot 2} = 91$ possibilities.

ii. Since no brand was ordered more than once, of the 91 non-negative solutions (b, w, v) to $b + w + v = 12$, we disallow those which are permutations of either $(11, 1, 0)$ or $(12, 0, 0)$. There are 6 of the former and 3 of the latter, hence $91 - 6 - 3 = 82$ possibilities in this case.

2. *Step 1:* The characteristic equation is $x^2 = 6x - 9$, which has the repeated root $x_1 = x_2 = 3$. Hence the general solution of the homogeneous equation is

$$a_{h,n} = (C_1 + C_2 n)3^n.$$

Step 2: Since 3 is a root of multiplicity two of the characteristic equation, whereas 1 is not a root at all, our choice of a particular solution should look like

$$a_{p,n} = a_{p_1,n} + a_{p_2,n},$$

where

$$a_{p_1,n} = C_3 \cdot n^2 \cdot 3^n, \quad a_{p_2,n} = C_4 n + C_5.$$

Inserting into the recurrence gives, firstly,

$$C_3 \cdot n^2 \cdot 3^n = 6C_3 \cdot (n-1)^2 \cdot 3^{n-1} - 9C_3 \cdot (n-2)^2 \cdot 3^{n-2} + 3^n.$$

The coefficients of $n^2 3^n$ will cancel exactly, as will those of $n3^n$. Comparing coefficients of 3^n yields

$$0 = C_3 \frac{6}{3} - C_3 \frac{9 \cdot 4}{9} + 1 \Rightarrow \cdots \Rightarrow C_3 = \frac{1}{2}.$$

Secondly,

$$C_4 n + C_5 = 6[C_4(n-1) + C_5] - 9[C_4(n-2) + C_5] + 4n - 4.$$

Comparing coefficients of n yields

$$C_4 = 6C_4 - 9C_4 + 4 \Rightarrow C_4 = 1$$

and comparing constant coefficients yields

$$C_5 = 6(C_5 - C_4) - 9(C_5 - 2C_4) + 4 \Rightarrow \dots (C_4 = 1) \dots \Rightarrow C_5 = 2.$$

Step 3: Hence, the general solution of the recurrence is

$$a_n = a_{h,n} + a_{p_1,n} + a_{p_2,n} = \left(C_1 + C_2 n + \frac{n^2}{2}\right) 3^n + n + 2.$$

Insert the initial conditions:

$$\begin{aligned} n = 0: \quad a_0 = 3 = C_1 + 2 \Rightarrow C_1 = 1, \\ n = 1: \quad a_1 = 8 = 3(C_1 + C_2) + \frac{3}{2} + (1+2) \Rightarrow \dots (C_1 = 1) \dots \Rightarrow C_2 = \frac{1}{6}. \end{aligned}$$

Hence,

$$a_n = \left(1 + \frac{n}{6} + \frac{n^2}{2}\right) 3^n + n + 2.$$

3. First some editing:

$$\begin{aligned} 2x \equiv 3 \pmod 7 \Rightarrow x \equiv 2^{-1} \cdot 3 \equiv 4 \cdot 3 \equiv 5 \pmod 7, \\ 3x \equiv 2 \pmod{11} \Rightarrow x \equiv 3^{-1} \cdot 2 \equiv 4 \cdot 2 \equiv 8 \pmod{11}, \\ 8x \equiv 7 \pmod{13} \Rightarrow x \equiv 8^{-1} \cdot 7 \equiv 5 \cdot 7 \equiv 35 \equiv 9 \pmod{13}. \end{aligned}$$

Thus, by eq. (11.3) in the lecture notes, the general solution is

$$x \equiv 5 \cdot b_1 \cdot 11 \cdot 13 + 8 \cdot b_2 \cdot 7 \cdot 13 + 9 \cdot b_3 \cdot 7 \cdot 11 \pmod{7 \cdot 11 \cdot 13}, \tag{1}$$

where

$$\begin{aligned} b_1 \equiv (11 \cdot 13)^{-1} \equiv ((-3) \cdot (-1))^{-1} \equiv 3^{-1} \equiv -2 \pmod 7, \\ b_2 \equiv (7 \cdot 13)^{-1} \equiv (7 \cdot 2)^{-1} \equiv 3^{-1} \equiv 4 \pmod{11}, \\ b_3 \equiv (7 \cdot 11)^{-1} \equiv (7 \cdot (-2))^{-1} \equiv (-1)^{-1} \equiv -1 \pmod{13}. \end{aligned}$$

We thus choose $b_1 = -2$, $b_2 = 4$, $b_3 = -1$ and insert into (1) to get

$$\begin{aligned} x &\equiv -5 \cdot 2 \cdot 11 \cdot 13 + 8 \cdot 4 \cdot 7 \cdot 13 - 9 \cdot 1 \cdot 7 \cdot 11 \\ &\equiv -1430 + 2912 - 693 \equiv 789 \pmod{1001}. \end{aligned}$$

Answer: The general solution is $x \equiv 789 \pmod{1001}$ and the largest negative solution is $x = 789 - 1001 = -212$.
Sanity Check: Check that $x = 789$ satisfies the original three congruences by direct calculation:

$789 - 5 = 784 = 7 \cdot 112$, ok

$789 - 8 = 781 = 11 \cdot 71$, ok

$789 - 9 = 780 = 13 \cdot 60$, ok !

4. (a) i. 17 is a prime so Fermat says that $3^{16} \equiv 1 \pmod{17}$. Thus

$$3^{158} = 3^{10 \cdot 16 - 2} \equiv (3^{16})^{10} \cdot 3^{-2} \equiv 1^{10} \cdot 3^{-2} \equiv (3^{-1})^2 \equiv 6^2 \equiv 36 \equiv 2 \pmod{17}.$$

ii. *Step 1:* Write the power in base 2.

$$\begin{aligned}158 &= 2\cdot 79+0,\\ 79 &= 2\cdot 39+1,\\ 39 &= 2\cdot 19+1,\\ 19 &= 2\cdot 9+1,\\ 9 &= 2\cdot 4+1,\\ 4 &= 2\cdot 2+0,\\ 2 &= 2\cdot 1+0,\\ 1 &= 2\cdot 0+1.\end{aligned}$$

Hence,

$$(158)_2 = 10011110 = 2^7+2^4+2^3+2^2+2^1.$$

Step 2: Repeated squaring. All congruences are modulo 17.

$$\begin{aligned}x_0\equiv 3,\quad x_1\equiv 3^2\equiv 9,\quad x_2\equiv 9^2\equiv 81\equiv -4,\\ x_3\equiv(-4)^2\equiv -1,\quad x_4\equiv(-1)^2\equiv 1,\quad x_5\equiv 1^2\equiv 1,\\ x_6\equiv 1^1\equiv 1,\quad x_7\equiv 1^1\equiv 1.\end{aligned}$$

Step 3: Pairwise multiplication, modulo 17.

$$3^{158}\equiv x_7x_4x_3x_2x_1\equiv 1\cdot 1\cdot(-1)\cdot(-4)\cdot 9\equiv 36\equiv 2,\ \text{ v.s.v.}$$

(b) We must check whether $3^k\equiv 1 \pmod{17}$ for any proper divisor k of 16. Since 16 is a power of two, we only need to check whether $3^8\equiv 1 \pmod{17}$. Modulo 17, the numbers 3^{2^j} are just the numbers x_j above. We've already computed $x_3\equiv -1$, thus 3 is a primitive root modulo 17.

5. *Step 1:* GCD(39, 52) = 13 so we can solve the Diophantine equation

$$39x+52y=13w \tag{2}$$

for any $w\in\mathbb{Z}$. Dividing across by 13, this is equivalent to the equation $3x+4y=w$. One solution is obviously $x=-w$, $y=w$ and so the general solution to (2) is

$$x=-w+4m,\quad y=w-3m,\quad m\in\mathbb{Z}. \tag{3}$$

Step 2: Solve $13w+84z=1$. We can do so since GCD(13, 84) = 1. Euclid forwards gives

$$\begin{aligned}84&=6\cdot 13+6,\\ 13&=2\cdot 6+1.\end{aligned}$$

Then backwards:

$$\begin{aligned}1=13-2\cdot 6&=13-2(84-6\cdot 13)\\ \Rightarrow 1&=13\cdot 13-2\cdot 84.\end{aligned}$$

So one solution is $w_0 = 13$, $z_0 = -2$. The general solution is

$$w = 13 - 84n, \quad z = -2 + 13n, \quad n \in \mathbb{Z}. \tag{4}$$

Step 3: Substituting the expression for w in (4) into (3) yields the full 2-parameter solution to the 3-variable equation:

$$x = -13 + 4m + 84n, \;\; y = 13 - 3m - 84n, \;\; z = -2 + 13n, \;\; m, n \in \mathbb{Z}.$$

6. (a) If we use Prim's algorithm, starting from s say, then it could proceed as follows:

Step	Edge chosen	Weight
1	$\{s, a\}$	6
2	$\{a, h\}$	5
3	$\{h, f\}$	2
4	$\{f, e\}$	2
5	$\{e, g\}$	2
6	$\{h, j\}$	3
7	$\{f, d\}$	3
8	$\{f, b\}$	3
9	$\{g, i\}$	3
10	$\{b, c\}$	4
11	$\{i, t\}$	6
		Total = 39

If we instead use Kruskal's algorithm, then it could proceed as follows, where Steps 1-3 are interchangebale, as are Steps 4-7 and Steps 10-11:

Step	Edge chosen	Weight
1	$\{h, f\}$	2
2	$\{f, e\}$	2
3	$\{e, g\}$	2
4	$\{h, j\}$	3
5	$\{f, d\}$	3
6	$\{f, b\}$	3
7	$\{g, i\}$	3
8	$\{b, c\}$	4
9	$\{b, a\}$	5
10	$\{a, b\}$	6
11	$\{i, t\}$	6
		Total = 39

Note that, with Prim, if we had instead chosen the edge $\{a, b\}$ in Step 2, we'd have ended up with the same MST as with Kruskal above. Similarly, if with Kruskal we'd instead chosen the edge $\{a, h\}$ in Step 10, we'd have ended up with the same MST as with Prim above.

(b)

Step	Edge chosen	Label assigned
1	$\{s, a\}$	$l(a) := 6$
2	$\{s, b\}$	$l(b) := 7$
3	$\{s, c\}$	$l(c) := 9$
4	$\{b, e\}$	$l(e) := 10$
5	$\{b, f\}$	$l(f) := 10$
6	$\{b, d\}$	$l(d) := 11$
7	$\{a, h\}$	$l(h) := 11$
8	$\{e, g\}$	$l(g) := 12$
9	$\{h, j\}$ or $\{f, j\}$	$l(j) := 14$
10	$\{e, i\}$	$l(i) := 14$
11	$\{i, t\}$ or $\{g, t\}$	$l(t) := 20$

In the above, Steps 4-5, 6-7 and 9-10 are all interchangeable. Depending on which edge is chosen in Step 11, there are two shortest paths of length 20, namely

$$s \to b \to e \to i \to t \quad \text{or} \quad s \to b \to e \to g \to t.$$

(c) The algorithm could proceed as follows (there are other alternatives). The squiggle in Step 6 means we decrease the flow along a backwards-directed arc.

Step	f-augmenting path	Increase in flow strength
1	$s \to a \to h \to j \to t$	3
2	$s \to b \to f \to j \to i \to t$	3
3	$s \to c \to g \to t$	6
4	$s \to a \to h \to f \to j \to i \to t$	1
5	$s \to b \to c \to e \to i \to t$	2
6	$s \to b \to c \to e \to i \rightsquigarrow j \to t$	2
7	$s \to c \to e \to g \to t$	2
Total flow strength		19

The final flow is illustrated in Figure L.6. The set of nodes that can now be reached from s via an augmenting path is $S = \{s, a, b, c, d, f, h\}$, where we note that b can only be reached by decreasing the flow along some backwards-directed arc. Set $T = V \backslash S = \{e, g, i, j, t\}$. Then we have

$$c(S, T) = c(h, j) + c(f, j) + c(c, e) + c(c, g) = 3 + 4 + 6 + 6 = 19 = |f|.$$

7. (a) See Definitions 5.7, 5.8 and 5.9 in the lecture notes.

(b) Theorem 5.10 in the lecture notes.

8. (a) See Dataset 21.1 and Definition 21.2 in the lecture notes.

(b) Theorem 21.3 in the lecture notes.

Figure 6

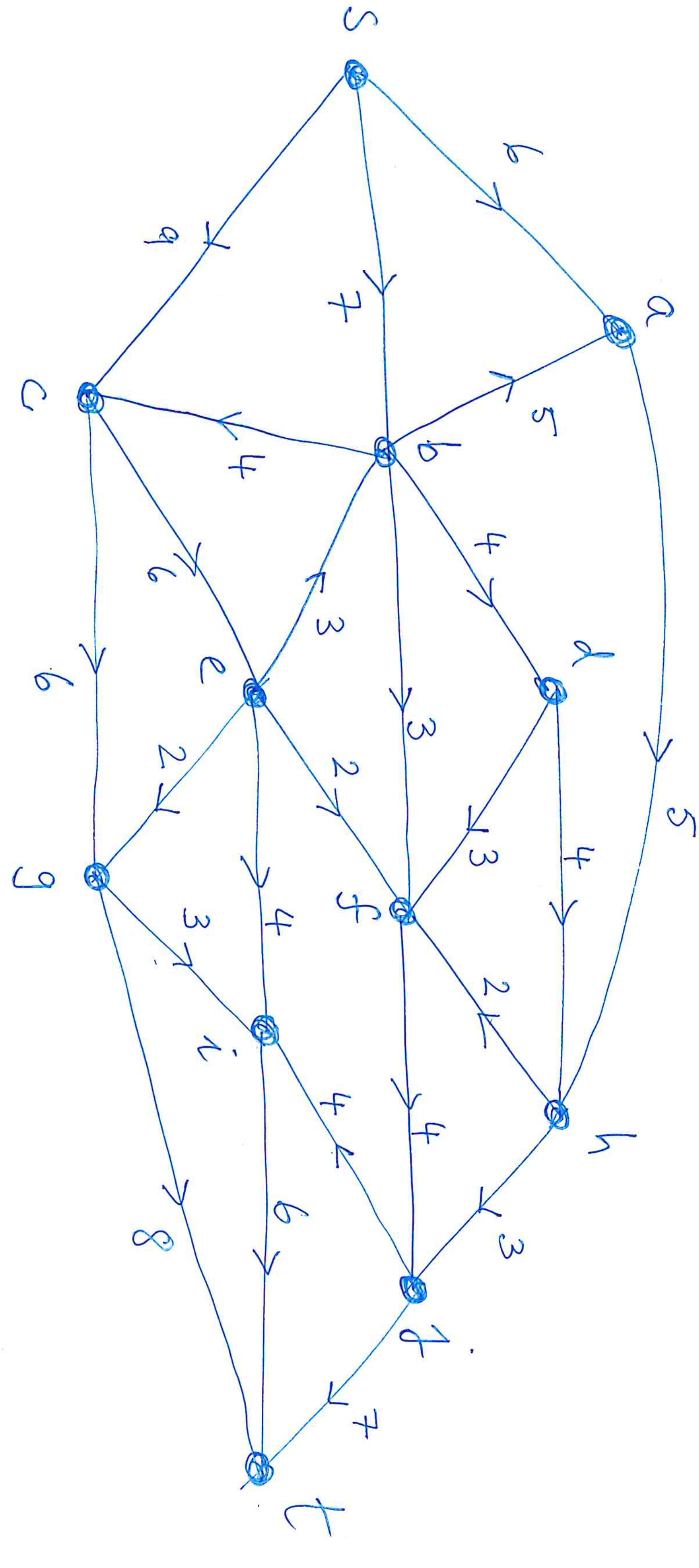

Figure L.6