Tentamen
MVE505 Diskret Matematik TM1/TM2

2022-08-24 kl. 14.00–18.00

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Hjälpmedel: Inga

För godkänt på tentan krävs 55 poäng, inklusive eventuella bonuspoäng erhållna från inlämningsuppgifterna under VT-2022. Preliminärt så krävs 73 poäng för betyget 4 och 90 poäng för betyget 5. Dessa gränser kan minskas men inte höjas i efterhand.

Lösningar läggs ut på kursens Canvassida direkt efter tentan. Tentan rättas och bedöms anonymt. Resultatet meddelas i Ladok senast den 14 september. Granskning ordnas därefter av kursansvrig.

OBS! Alla stegen i dina resonemang måste motiveras väl i skrift och alla beräkningar visas. Det är i huvudsak tillvägagångssätten och motiveringarna som ger poäng, inte svaren.

I de uppgifter som består av fler olika delar går det alltid att lösa de enskilda delarna oberoende av varandra, även om man kan ibland spara räknetid genom att lösa deluppgifterna sekventiellt.

Om du i en lösning av någon av uppgifterna 1-5 åberopar en sats från kurslitteraturen så behöver du *inte* inkludera ett bevis av satsen. I uppgift 1 behöver du *inte* ange svaren som explicita decimaltal.

Jag bifogar 3 exemplar av Figur 4.1 så att ni har extra kladdpapper.

Var god vänd!

Uppgifterna

1. Every Saturday Kalle goes to Willys and buys the same 16 grocery items, which for simplictity we'll call A-P. In May his total bill was 600kr but, because of rampant inflation, last week his bill was 700kr. Assuming the price of each item is an integer and that no item is free,

(a) How many possibilities are there for Kalles kvitto in May (the kvitto shows the price of each item) ? (3p)

(b) Assuming we have knowledge of his kvitto from May, how many possibilities would this leave for his most recent kvitto, assuming none of the 16 items has gotten strictly cheaper ? (3p)

(c) Item C on Kalles list is "cranberry juice". Considering this as one word ("cranberryjuice"), how many 14-letter "words" (ie.: arbitrary strings of letters) can one make from its letters

i. in total ? (3p)

ii. in which *all* the vowels are adjacent but not *all* the consonants ? (3p)

iii. which neither start with c nor end with e ? (3p)

2. Determine an explicit formula for the numbers $(a_n)_{n=0}^{\infty}$ which satisfy the recursion (12p)

$$a_0 = 1, \;\; a_1 = 3, \quad a_{n+2} = 2a_{n+1} - a_n + 2^n + 1, \;\; \forall\, n \geq 0.$$

3. (a) Find the general solution of the congruence (6p)

$$74x \equiv 6 \pmod{134}.$$

(b) Compute $5^{47} \pmod{18}$

i. using Euler's theorem (4p)

ii. using the repeated squaring algorithm. (4p)

Express your answer as an integer in the interval [0, 17].

(Obs! Here I want you to show that you understand how both these techniques work. Hence, you will get *zero* points for any other solution, even if your answer is correct.)

Var god vänd!

4. (a) Let G be the network in Figure 4.1 and let G^* be the underlying, undirected and unweighted graph.

   i. Consider the matching

$$M = \{\{s, b\}, \{d, e\}, \{f, g\}, \{a, t\}\}.$$

   Determine an M-augmenting path which includes every edge of M and thereby replace M by a maximum matching in G^*. (3.5p)

   ii. Apply the Ford-Fulkerson algorithm to determine a maximum flow from s to t and a corresponding minimum cut in G. Write clearly which f-augmenting path you choose at each step. Draw the final flow in full and indicate the corresponding minimum cut. (5.5p)

   (b) Apply the Gale-Shapley algorithm, with X as the set of proposers, to find a stable matching for the data in Figure 4.2. Write clearly which proposals and rejections are issued in each round and the state of the strings after each round. (6p)

5. Prove that if p is a prime then $(p-1)! \equiv -1 \pmod{p}$. (7p)

   (Tips: Think about inverses mod p.).

6. (a) For a real number t and a positive integer k, define the *generalised binomial coefficient* $\binom{t}{k}$. (1p)

   (b) For t a negative integer, say $t = -n$, express $\binom{t}{k}$ in terms of a "normal" binomial coefficient (i.e.: one in which the parameter t is also a positive integer) (2p)

   (c) State and prove *combinatorially* the binomial theorem for negative integer exponents. (9p)

7. Let n, k be positive integers.

   (a) Explain what is meant by a *k-cycle* $\sigma \in S_n$. (1p)

   (b) Define what is meant by the Stirling number $s(n, k)$ of the *first* kind. (1p)

   (c) Write down and prove a recursion satisfied by the numbers $s(n, k)$. (10p)

8. (a) Describe Prim's algorithm for finding a minimum spanning tree (MST) in a weighted, connected graph. (2p)

   (b) Prove that Prim's algorithm always yields a MST. (11p)

Go n'eirí an bóthar libh!

Lösningar Diskret Matematik TM1/TM2, 220824

1. (a) i. Let x_i be the price in kronor of item i, $i = 1, \ldots, 16$. Then each $x_i \in \mathbb{Z}_+$ and $\sum_{i=1}^{16} x_i = 600$. Let $y_i = x_i - 1$, so each y_i is a non-negative integer and $\sum_{i=1}^{16} y_i = 584$. The number of possibilities for the receipt equals the number of solutions to this equation which, by Example 1.13 in the lecture notes, is $\binom{584+16-1}{16-1} = \binom{599}{15}$.

ii. Let z_i be the amount by which item i has increased in price since May. Then each z_i is a non-negative integer and $\sum_{i=1}^{16} z_i = 100$. The number of solutions equals $\binom{100+16-1}{16-1} = \binom{115}{15}$.

(b) i. There are 3 r's, 2 c's and 2 e's. Hence, by Theorem 2.6, the number of possible words is $\frac{14!}{3!\,2!\,2!}$.

ii. There are 5 vowels, of which 2 are e's. If they are to be all adjacent but not all the consonants, then the vowels must be placed in positions $2-6$ or $3-7$ or ... or $9-13$. There are thus 8 possibilities for where to place the vowels. Having chosen where to place them there are, by Theorem 2.6, $\frac{5!}{2!}$ possible ways to permute them and then $\frac{9!}{3!\,2!}$ ways to permute the remaining consonants.
By MP, the total number of possible words is $8 \times \frac{5!}{2!} \times \frac{9!}{3!\,2!}$.

iii. Let X denote the set of all possible words (in part (i)) and let A_c (resp. A_e) denote the subset of X consisting of those words which begin with a c (resp. which end with an e). We seek $|X \backslash (A_c \cup A_e)|$. By Inclusion-Exclusion,

$$|X \backslash (A_c \cup A_e)| = |X| - |A_c| - |A_e| + |A_c \cap A_e|. \tag{1}$$

We have already seen in (i) that

$$|X| = \frac{14!}{3!\,2!\,2!}. \tag{2}$$

Next consider A_c. If a word starts with c, then there remain to append 13 letters, of which 2 are r's and 2 are e's. Hence,

$$|A_c| = \frac{13!}{3!\,2!}. \tag{3}$$

Similarly, if a word ends with e, then there remain to prepend 13 letters of which 3 are r's and 2 are c's. So also

$$|A_e| = \frac{13!}{3!\,2!}. \tag{4}$$

Finally, if a word starts with c and ends with e, then there remain to permute 12 letters of which 3 are r's. Thus,

$$|A_c \cap A_e| = \frac{12!}{3!}. \tag{5}$$

Substituting (2)-(5) into (1) gives our answer:

$$|X \backslash (A_c \cup A_e)| = \frac{14!}{3!\,2!\,2!} - \frac{13!}{3!} + \frac{12!}{3!}.$$

2. *Step 1:* The characteristic equation is $x^2 = 2x - 1$, which has the repeated root $x_1 = x_2 = 1$. Hence the general solution of the homogeneous equation is

$$a_{h,n} = C_1 + C_2 n.$$

Step 2: Since 1 is a root of multiplicity two of the characteristic equation, whereas 2 is not a root at all, our choice of a particular solution should look like

$$a_{p,n} = a_{p_1,n} + a_{p_2,n},$$

where

$$a_{p_1,n} = C_3 \cdot 2^n, \quad a_{p_2,n} = C_4 n^2.$$

Inserting into the recurrence gives, firstly,

$$C_3 \cdot 2^{n+2} = 2C_3 \cdot 2^{n+1} - C_3 \cdot 2^n + 2^n \Rightarrow \cdots \Rightarrow C_3 = 1.$$

Secondly,

$$C_4(n+2)^2 = 2C_4(n+1)^2 - C_4 n^2 + 1.$$

The coefficients of both n^2 and n will cancel exactly. Comparing constant coefficients yields

$$4C_4 = 2C_4 + 1 \Rightarrow C_4 = \frac{1}{2}.$$

Step 3: Hence, the general solution of the recurrence is

$$a_n = a_{h,n} + a_{p_1,n} + a_{p_2,n} = C_1 + C_2 n + 2^n + \frac{n^2}{2}.$$

Insert the initial conditions:

$$\begin{aligned} n = 0: \quad a_0 = 1 = C_1 + 1 &\Rightarrow C_1 = 0, \\ n = 1: \quad a_1 = 3 = C_1 + C_2 + \frac{1}{2} + 2 &\Rightarrow C_2 = \frac{1}{2}. \end{aligned}$$

Hence,

$$a_n = 2^n + \frac{n^2}{2} + \frac{n}{2}.$$

3. (a) Först Euklides framåt:

$$\begin{aligned} 134 &= 1 \cdot 74 + 60, \\ 74 &= 1 \cdot 60 + 14, \\ 60 &= 4 \cdot 14 + 4, \\ 14 &= 3 \cdot 4 + 2, \\ 4 &= 2 \cdot 2 + 0. \end{aligned}$$

Så SGD(134, 74) = 2 och $2 \mid 6$ så kongruensen har en lösning. Vi kan dela igenom med 2 för att få den ekvivalenta kongruensen

$$37x \equiv 3 \pmod{67},$$

vars allmänna lösning är

$$x \equiv 37^{-1} \cdot 3 \pmod{67}. \tag{6}$$

För att beräkna inversen så går vi bakåt igenom Euklides, där jag delar allting med 2:

$$\begin{aligned}
1 &= 7 - 3 \cdot 2 \\
&= 7 - 3(30 - 4 \cdot 7) \\
&= 13 \cdot 7 - 3 \cdot 30 \\
&= 13(37 - 30) - 3 \cdot 30 \\
&= 13 \cdot 37 - 16 \cdot 30 \\
&= 13 \cdot 37 - 16(67 - 37) \\
\Rightarrow 1 &= -16 \cdot 67 + 29 \cdot 37.
\end{aligned}$$

Vi läser detta modulo 67, vilket innebär att $37^{-1} \equiv 29 \pmod{67}$. Insättning in i (6) medför att

$$x \equiv 29 \cdot 3 \equiv 20 \pmod{67}.$$

(b) i. $18 = 2 \cdot 3^2$ så $\phi(18) = \phi(2)\phi(3^2) = (2-1)(3^2-3) = 6$. Eftersom $\text{SGD}(5, 18) = 1$ innebär Eulers sats att $5^6 \equiv 1 \pmod{18}$. Således är $5^{47} = (5^6)^8 \cdot 5^{-1} \equiv 5^{-1}$. Men $5 \cdot 11 = 55 \equiv 1 \pmod{18}$ så $5^{-1} \equiv 11 \pmod{18}$.

ii. Först skriver vi potensen i bas 2. Vi har

$$\begin{aligned}
&47 = 2 \cdot 23 + 1, \\
&23 = 2 \cdot 11 + 1, \\
&11 = 2 \cdot 5 + 1, \\
&5 = 2 \cdot 2 + 1, \\
&2 = 2 \cdot 1 + 0, \\
&1 = 2 \cdot 0 + 1,
\end{aligned}$$

vilket innebär att $47 = (101111)_2 = 2^5 + 2^3 + 2^2 + 2^1 + 2^0$.
Sätt $x_0 := 5^{2^0} = 5 \pmod{18}$ och beräkna successivt

$$\begin{aligned}
x_1 \equiv 5^2 &\equiv 7, \quad x_2 \equiv 7^2 \equiv -5, \\
x_3 \equiv (-5)^2 &\equiv 7, \quad x_4 \equiv 7^2 \equiv -5, \\
&x_5 \equiv (-5)^2 \equiv 7.
\end{aligned}$$

Således har vi att

$$\begin{aligned}
5^{47} &\equiv x_5 x_3 x_2 x_1 x_0 \equiv (7 \cdot 7) \cdot (-5) \cdot 7 \cdot 5 \\
&\equiv ((-5) \cdot (-5)) \cdot 7 \cdot 5 \equiv (7 \cdot 7) \cdot 5 \equiv (-5) \cdot 5 \equiv -7 \equiv 11,
\end{aligned}$$

vilket stämmer överens med svaret vi fick från Eulers sats.

4. (a) i. An M-augmenting path must start and end at an unmatched node and the only such nodes are c and h. An M-augmenting path starting at c and ending at h, and including all edges of M, is

$$c \to s \rightsquigarrow b \to a \rightsquigarrow t \to g \rightsquigarrow f \to d \rightsquigarrow e \to h.$$

Exchanging edges along the path gives the maximum matching

$$M^* = \{\{c,\, s\},\, \{b,\, a\},\, \{t,\, g\},\, \{f,\, d\},\, \{e,\, h\}\}.$$

ii. The algorithm could proceed as follows (there are other alternatives). Note that in each of the last two steps we decrease flow along a backwards oriented arc.

Step	f-augmenting path	Increase in flow strength
1	$s \to a \to t$	8
2	$s \to b \to d \to f \to t$	5
3	$s \to c \to b \to e \to g \to t$	4
4	$s \to e \to h \to g \to t$	3
5	$s \to e \to f \to t$	3
6	$s \to b \to f \to g \to t$	2
7	$s \to c \to b \to f \to g \to t$	1
8	$s \to e \to f \to g \to t$	1
9	$s \to e \to f \rightsquigarrow b \to a \to t$	1
10	$s \to e \rightsquigarrow b \to a \to t$	2
Total flow strength		30

The final flow is illustrated in Figure L.4. The set of nodes that can now be reached from s via an augmenting path is $S = \{s,\, c\}$. Sätt $T = V \backslash S = \{a,\, b,\, d,\, e,\, f,\, g,\, h,\, t\}$. Then we have

$$c(S,\, T) = c(s,\, a) + c(s,\, b) + c(c,\, b) + c(s,\, e) = 8 + 7 + 5 + 10 = 30 = |f|.$$

(b)

Round	Proposals	Rejections	Strings
1	$(\alpha,\, A)$, $(\beta,\, A)$, $(\gamma,\, B)$, $(\delta,\, A)$	$(A,\, \alpha)$, $(A,\, \beta)$	$(A,\, \delta)$, $(B,\, \gamma)$
2	$(\alpha,\, B)$, $(\beta,\, C)$	$(B,\, \gamma)$	$(A,\, \delta)$, $(B,\, \alpha)$, $(C,\, \beta)$
3	$(\gamma,\, C)$	$(C,\, \beta)$	$(A,\, \delta)$, $(B,\, \alpha)$, $(C,\, \gamma)$
4	$(\beta,\, D)$		$(A,\, \delta)$, $(B,\, \alpha)$, $(C,\, \gamma)$, $(D,\, \beta)$

5. Since p is a prime, each number $x \in [1,\, p-1]$ has a multiplicative inverse mod p. Hence, in the product $(p-1)! = 1 \cdot 2 \cdot \cdots \cdot (p-1)$, we can pair off the terms such that the product of each pair is 1 (mod p). The only numbers which are their own inverses are 1 and $p-1$. Hence, mod p, we'll be left with $(p-1)! \equiv 1 \cdot \cdots \cdot 1 \cdot (p-1) \equiv -1 \pmod p$, v.s.v.

6. (a)

$$\binom{t}{k} = \frac{t(t-1) \cdot \cdots \cdot (t-k+1)}{k!}.$$

(b)

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

(c) Theorem 5.1 in the lecture notes.

7. (a) Definition 13.1 in the lecture notes.

(b) Definition 13.7 in the lecture notes.

(c) Theorem 13.8 in the lecture notes.

8. (a) See Lecture Notes 18.

(b) Theorem 18.8 in the lecture notes.

Figure 4.1

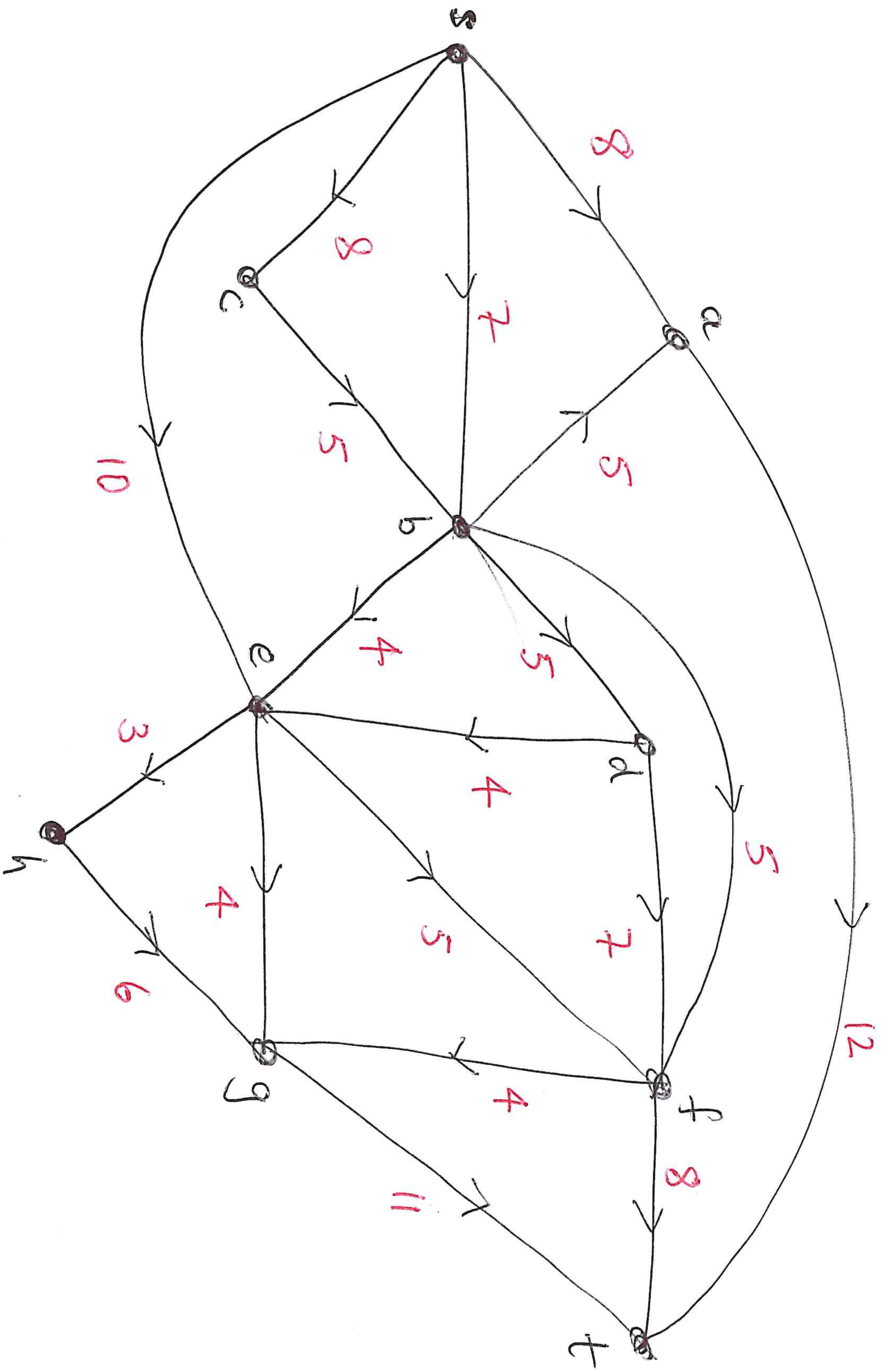

Figure 4.2

	A	B	C	D
α	1, 3	2, 1	3, 4	4, 3
β	1, 4	4, 3	2, 3	3, 1
γ	3, 2	1, 4	2, 2	4, 2
δ	1, 1	2, 2	4, 1	3, 4

$$X = \{\alpha, \beta, \gamma, \delta\}$$

$$Y = \{A, B, C, D\}$$

Figur L.4