# Tentamen MVE505 Diskret Matematik TM2

### 2022-03-15 kl. 14.00-18.00

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### Hjälpmedel: Inga

För godkänt på tentan krävs 55 poäng, inklusive eventuella bonuspoäng erhållna från inlämningsuppgifterna under VT-2022. Preliminärt så krävs 73 poäng för betyget 4 och 90 poäng för betyget 5. Dessa gränser kan minskas men inte höjas i efterhand.

Lösningar läggs ut på kursens Canvassida direkt efter tentan. Tentan rättas och bedöms anonymt. Resultatet meddelas i Ladok senast den 5 april. Granskning ordnas därefter av kursansvrig.

**OBS!** Alla stegen i dina resonemang måste motiveras väl i skrift och alla beräkningar visas. Det är i huvudsak tillvägagångssätten och motiveringarna som ger poäng, inte svaren.

I de uppgifter som består av fler olika delar går det alltid att lösa de enskilda delarna oberoende av varandra, även om man kan ibland spara räknetid genom att lösa deluppgifterna sekventiellt.

Om du i en lösning av någon av uppgifterna 1-5 åberopar en sats från kurslitteraturen så behöver du *inte* inkludera ett bevis av satsen. I uppgift 1 behöver du *inte* ange svaren som explicita decimaltal.

## Uppgifterna

- 1. At the moment I started writing this exercise, 67 of 227 countries in the Worldometers database had reported their new Covid case totals for the day, giving a total of 684656 new cases for the day. (OBS! Each reporting country has reported *at least one* new case.)
  - (a) How many possibilities does this give for the current list of new cases by country
    - i. if you know which 67 countries have reported ?
    - ii. if you don't know this ?
  - (b) For each country the number of new cases has either gone up, gone down or stayed (5p) the same as yesterday. If all you're interested in is the trend (up, down or =) and you know which 67 countries have reported
    - i. how many possibilities are there for the current list of trends by country ?
    - ii. how many possibilities are there if I told you that 40 countries have seen a decrease since yesterday, 25 have seen an increase and 2 have stayed the same ?
  - (c) One of the countries in the world is BRUNEI. How many "words" (i.e.: 6-letter strings) (4p) can you make of the letters in BRUNEI which don't contain any adjacent vowels ?

Var god vänd!

(5p)

2. Solve the recurrence

 $a_0 = a_1 = 1$ ,  $a_n = 3a_{n-1} + 10a_{n-2} + 5^n + 12$ ,  $\forall n \ge 2$ .

**3**. (a) i. For which  $b \in \mathbb{Z}$  does the system of congruences

 $84x \equiv b \pmod{49}, \quad 2x \equiv 3 \pmod{11}, \quad x \equiv 2 \pmod{9}$ 

have a solution ?

- ii. Pick any such b (your choice !) and find the general solution of the system in that case.
- (b) Compute  $17^{2687} \pmod{5220}$ . Give your answer as a number in [0, 5219]. (6p) (HINT:  $307 \cdot 17 = 5219$ ).
- 4. Let G be the weighted, directed graph in Figure 4. Let  $G^*$  be the underlying undirected, but still weighted graph and let  $G^{**}$  be the underlying undirected and unweighted graph.
  - (a) Determine in  $G^{**}$ 
    - i. an Euler trail or circuit (whichever exists)
    - ii. a Hamilton cycle
    - iii. a maximum matching
    - iv. an optimal coloring (motivate why your coloring is optimal !).
  - (b) Apply either Prim's or Kruskal's algorithm (your choice !) to determine a MST in (4p)  $G^*$ . Indicate clearly which edge you choose in each step and the final tree.
  - (c) Apply the Ford-Fulkerson algorithm to determine a maximum flow in G from s to t (5p) along with a minimum cut. Indicate clearly which f-augmenting path you choose at each step. Draw the final flow and cut.
- 5. Let G be a finite group whose size is a power of a prime. Prove that the center Z(G) must (7p) be a non-trivial subgroup.
- 6. (a) Define the term *Dyck path* and thereby (or otherwise) define the *Catalan numbers* (2p)  $(C_n)_{n=0}^{\infty}$ .
  - (b) Prove that  $C_n = \frac{1}{n+1} {\binom{2n}{n}}.$  (11p)
- 7. (a) Define what is meant by a (right) action of a group G on a set S. (2p)
  (b) State and prove Burnside's Lemma for the number of orbits of an action by a finite group G on a finite set S. (11p)
- 8. (a) Define the term *stable matching* for a bipartite dataset. Your definition should include (2p) a clear explanation of what the dataset consists of.
  - (b) Prove that a stable matching always exists for a bipartite dataset. (11p)

#### Go n'eirí an bóthar libh!

(8p)

(6p)

#### Lösningar Diskret Matematik TM2, 220315

- 1. (a) i. We have to place 684656 identical balls (the new cases) into 67 distinguishable bins (the reporting countries) in such a way that each bin gets at least one ball. So first place one ball in each bin, leaving 684656 67 balls to be distributed freely. The number of possibilities, by Proposition 1.11, is  $\binom{(684656-67)+67-1}{67-1} = \binom{684655}{66}$ .
  - ii. Firstly, there are  $\binom{227}{67}$  possibilities for the reporting countries. Once these are chosen, the number of possibilities is the same as before. Hence, the answer is now  $\binom{227}{67} \cdot \binom{684655}{66}$ .
  - (b) i. For each country there are 3 possible trends: up, down or =. Hence, by MP, there are  $3^{67}$  possibilities for the list of trends.
    - ii. You have to choose which 40 countries are down, which 25 are up and which 2 are =. By Theorem 2.6, the number of possibilities is  $\frac{67!}{40!\,25!\,2!}$ .
  - (c) If there are to be no adjacent vowels, then the vowels must be placed in one of the following sets of three positions: (1, 3, 5), (1, 3, 6), (1, 4, 6), (2, 4, 6). So there are 4 possibilities for the positions of the vowels. Having positioned them, there are 3! = 6 possible internal orderings of them, and also 3! = 6 internal orderings of the remaining three consonants.

In summary, by AP and MP, the number of possible words is  $4 \cdot 6 \cdot 6 = 144$ .

Alterntive Solution: Use Inclusion-Exclusion. Let X be the set of all possible words that can be made from the 6 letters. Let  $A_1$ ,  $A_2$ ,  $A_3$  consist respectively of those words with U & E, U & I resp. E & I adjacent. We seek

$$|X \setminus (A_1 \cup A_2 \cup A_3)| = |X| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$$
(1)

Firstly, we have |X| = 6!.

Secondly, if U & E are adjacent, then they can form either UE or EU. In either case, we consider this as one letter and thus have effectively 5 letters we can permute freely. So  $|A_1| = 2 \cdot 5!$ . By symmetry,  $A_2$  and  $A_3$  have the same size.

Thirdly, if U & E are adjacent and also E & I, then the three letters combined must form either UEI or IEU. In either case, we consider the combination as one letter, effectively giving 4 letters we can permute freely. Thus  $|A_1 \cap A_2| = 2 \cdot 4!$ . By symmetry,  $A_1 \cap A_3$  and  $A_2 \cap A_3$  have the same size.

Fourthly, it is not possible to have all three pairs of vowels adjacent, so  $|A_1 \cap A_2 \cap A_3| = 0$ .

Substituting everything into (1) yields

$$|X \setminus (A_1 \cup A_2 \cup A_3)| = 6! - 3(2 \cdot 5) + 3(2 \cdot 4!) - 0 = \dots = 144.$$

**2**. Step 1: The characteristic equation is  $x^2 = 3x + 10$ , which has roots  $x_1 = 5$ ,  $x_2 = -2$ . Hence the general solution of the homogeneous equation is

$$a_{h,n} = C_1 \cdot 5^n + C_2 \cdot (-2)^n.$$

Step 2: Since 5 is a root of multiplicity one of the characteristic equation, whereas 1 is not a root at all, our choice of a particular solution should look like

$$a_{p,n} = a_{p_1,n} + a_{p_2,n},$$

where

$$a_{p_1,n} = C_3 \cdot n \cdot 5^n, \quad a_{p_2,n} = C_4.$$

Inserting into the recurrence gives, firstly,

$$C_4 = 3C_4 + 10C_4 + 12 \Rightarrow C_4 = -1.$$

Secondly,

$$C_3 n 5^n = C_3(n-1)5^{n-1} + C_3(n-2)5^{n-2} + 5^n$$

The coefficients of  $n5^n$  will cancel exactly and comparing coefficients of  $5^n$  yields

$$0 = -\frac{3C_3}{5} - \frac{4C_3}{5} + 1 \Rightarrow C_3 = \frac{5}{7}.$$

Step 3: Hence, the general solution of the recurrence is

$$a_n = a_{h,n} + a_{p_1,n} + a_{p_2,n} = C_1 \cdot 5^n + C_2 \cdot (-2)^n + \frac{n5^{n+1}}{7} - 1.$$

Insert the initial conditions:

$$n = 0: \quad a_0 = 1 = C_1 + C_2 - 1 \Rightarrow C_1 + C_2 = 2 = \frac{14}{7},$$
  
$$n = 1: \quad a_1 = 1 = 5C_1 - 2C_2 + \frac{25}{7} - 1 \Rightarrow 5C_1 - 2C_2 = 2 - \frac{25}{7} = -\frac{11}{7}.$$

Solving yields  $C_1 = 17/49$ ,  $C_2 = 81/49$  and hence

$$a_n = \left(\frac{17+35n}{49}\right)5^n + \frac{81}{49}(-2)^n - 1.$$

- **3**. (a) i. Since 49, 11 and 9 are pairwise relatively prime, there is no problem on that front. However, GCD(84, 49) = 7 so b must be a multiple of 7.
  - ii. We take b = 7 and first simplify the system of congruences:

$$84x \equiv 7 \pmod{49} \Leftrightarrow 12x \equiv 1 \pmod{7} \Rightarrow x \equiv (12)^{-1} \equiv 5^{-1} \equiv 3 \pmod{7}.$$
$$2x \equiv 3 \pmod{11} \Rightarrow x \equiv 2^{-1} \cdot 3 \equiv 6 \cdot 3 \equiv 7 \pmod{11}.$$
$$x \equiv 2 \pmod{9}.$$

By (11.3), the general solution is

$$x \equiv 3 \cdot b_1 \cdot 11 \cdot 9 + 7 \cdot b_2 \cdot 7 \cdot 9 + 2 \cdot b_3 \cdot 7 \cdot 11 \pmod{7 \cdot 11 \cdot 9},$$
(2)

where

$$b_1 \equiv (11 \cdot 9)^{-1} \equiv (4 \cdot 2)^{-1} \equiv 8^{-1} \equiv 1^{-1} \equiv 1 \pmod{7},$$
  

$$b_2 \equiv (7 \cdot 9)^{-1} \equiv 63^{-1} \equiv (-3)^{-1} \equiv -4 \pmod{11},$$
  

$$b_3 \equiv (7 \cdot 11)^{-1} \equiv ((-2) \cdot 2)^{-1} \equiv 5^{-1} \equiv 2 \pmod{9}.$$

Substituting into (2) gives

$$x \equiv 3 \cdot 1 \cdot 11 \cdot 9 - 4 \cdot 7 \cdot 7 \cdot 9 + 2 \cdot 2 \cdot 7 \cdot 11 \equiv 297 - 1764 + 308 \equiv -1159 \equiv 227 \pmod{693}.$$

(b) First factorise:

$$5220 = 2 \cdot 2610 = 2^2 \cdot 1305 = 2^2 \cdot 3 \cdot 435 = 2^2 \cdot 3^2 \cdot 145 = 2^2 \cdot 3^2 \cdot 5 \cdot 29.$$

In particular, GCD(17, 5220) = 1 so Euler's Theorem applies. We have

$$\Phi(5220) = \Phi(2^2) \cdot \Phi(3^2) \cdot \Phi(5) \cdot \Phi(29) = (2^2 - 2^1) \cdot (3^2 - 3^1) \cdot (5 - 1) \cdot (29 - 1) = 1344$$

and thus

$$17^{2687} = 17^{2 \cdot 1344 - 1} \equiv (17^{1344})^2 \cdot 17^{-1} \equiv 1^2 \cdot 17^{-1} \equiv 17^{-1} \pmod{5220}.$$

But the hint tells us that  $307 \cdot 17 \equiv -1 \Rightarrow 17^{-1} \equiv -307 \equiv 4913 \pmod{5220}$ .

4. (a) i. Nodes f and t have odd degree, while every other node has even degree. An example of an Euler trail between f and t is

$$\begin{split} f \to c \to s \to e \to a \to s \to b \to a \to d \to b \to c \\ \to d \to f \to e \to d \to g \to f \to t \to e \to g \to t. \end{split}$$

ii. For example,

 $s \to a \to e \to t \to f \to g \to d \to c \to b \to s.$ 

iii. For example,

 $M = \{\{s, a\}, \{b, c\}, \{d, e\}, \{f, g\}\}.$ 

Note that one node must be left unmatched since  $G^{**}$  has an odd number (9) of nodes.

iv.  $\chi(G^{**}) = 4$ . We need at least 4 colors, for example becuase of the  $W_5$  centered at d with spokes out to a, e, f, c, b. If we apply the greedy algorithm starting with s, then in alphabetical order before finishing with t, we get the following 4-coloring:

Color 1: s, c, g; Color 2: a, f; Color 3: b, e; Color 4: d, t.

Step	Edge chosen	Weight	
1	$\{s, b\}$	5	
2	$\{b, a\}$	2	
3	$\{b, c\}$	4	
4	$\{c, f\}$	2	
5	$\{f, e\}$	3	
6	$\{c, d\}$	3	
7	$\{d,g\}$	4	
8	$\{e, t\}$	6	
Total weight		29	

(b) If we use Prim's algorithm starting at s, say, then it could proceed as follows:

If we instead used Kruskal's algorithm, it could proceed as follows (there are different options in several steps, but the choice below illustates that the MST for this graph is not unique):

Step	Edge chosen	Weight
1	$\{a, b\}$	2
2	$\{c, f\}$	2
3	$\{d,f\}$	3
4	$\{e, f\}$	3
5	$\{a, d\}$	4
6	$\{d,g\}$	4
7	$\{s, b\}$	5
8	$\{e, t\}$	6
Total weight		29

Step	f-augmenting path	Increase in flow strength
1	$s \to b \to e \to t$	6
2	$s \to a \to e \to g \to t$	5
3	$s \to b \to d \to g \to t$	3
4	$s \to c \to f \to t$	2
5	$s \to c \to d \to f \to t$	3
6	$s \to b \to d \to g \to f \to t$	1
Total flow strength		20

(c) An example of how the algorithm might proceed is as follows (there are other alternatives):

The resulting maximum flow is illustrated in Figure S.4. The set of nodes reachable from s by an augmenting path is then  $S = \{s, a, b, c, d\}$ . Set  $T = V \setminus S = \{e, f, g, t\}$ . Then

$$c(S, T) = c(s, e) + c(a, e) + c(d, g) + c(d, f) + c(c, f) = 6 + 5 + 4 + 3 + 2 = 20 = |f|.$$

**5**. Use the class equation for G in the form (15.6):

$$|G| = |Z(G)| + \sum_{g_i \notin Z(G)} \frac{|G|}{|C_G(g_i)|}.$$
(3)

For each  $g_i \notin Z(G)$ ,  $C_G(g_i)$  is a subgroup of G. Hence, by Lagrange's theorem, its size is a divisor of |G|. Since |G| is a power of some prime, p say, then so must be  $|C_G(g_i)|$  and hence  $|G|/|C_G(g_i)|$  must be divisible by p (since if the quotient were  $p^0 = 1$ , it would mean that  $g_i \in Z(G)$ ). Hence every term in the sum in (3) is divisible by the prime p, as is the LHS. Thus the term |Z(G)| must also be a multiple of p. In particular, |Z(G)| > 1, v.s.v.

- 6. (a) Definitions 5.7, 5.8 and 5.9 in the notes.
  - (b) Theorem 5.10 in the notes.
- 7. (a) Definition 14.1 in the notes.
  - (b) Theorem 14.14 in the notes.
- 8. (a) Dataset 21.1 and Definition 21.2 in the notes.
  - (b) Theorem 21.3 in the notes.

Figure 4



Figure S.4

