Sannolikhet, statistik och risk MVE240
Written examination - 19th August 2009, 8.30-12.30.

Allowed aids: The course book "Probability and Risk Analysis - An Introduction for Engineers", earlier versions of the book (compendium). Mathematical or statistical tables. The tables of formulæ of the course. Any calculator (not PC). Dictionaries for translation.
Jour: Igor Rychlik anknytning 3545 (examinator).

1 Suppose that (on a certain roof) the maximal snow load under one year is Gumbel distributed, i.e.

$$F_S(s) = e^{-e^{-\frac{s-13}{1.5}}}.$$

Determine the so-called 50-years snow load. (10 p)

2 The life time of cars are studied. By life time we mean the meter reading (mätarställning) when the car is scrapped (skrotad). The life time of a large number of cars in Göteborg are checked. The obtained failure intensity of a single car is

$$\lambda(x) = 5 \cdot 10^{-7} + (3 \cdot 10^{-13}) \cdot x^2$$

when the meter reading is x (unit: Swedish mile). Calculate the probability the meter reading exceeds 15 000 miles when the car is scrapped. (10 p)

3 The random variable X describes the strength of a component. It is Weibull distributed; the characteristic strength is 10 kN and the coefficient of variation R_X is 0.2. The characteristic strength is defined as the 0.9-quantile. The Weibull distribution has two parameters a and c,

$$F_X(x) = 1 - e^{-(x/a)^c}.$$

The component will be exposed to a load equal to 8 kN. Calculate the probability, $P(X < 8)$. Hint: Use the table below. (10 p)

c	2.00	2.10	2.70	3.00	3.68	4.00	5.00	5.73	8.00
R_X	0.523	0.500	0.400	0.363	0.300	0.281	0.220	0.200	0.118

4 From record of repairs of some offshore equipments, it is found that the failure free operation time T (that is, time between breakdowns) of an equipment is lognormally distributed, with mean of 6 months and coefficient of variation 0.25. The safety requirement is that there should be at least 90% probability that a piece of equipment will be operational at any time. (The rule is not easy to interpret.) The problem continues on the next sheet.

Suppose that maintenance of the equipment restores its original lifetime, i.e. after maintenance an equipment is as reliable as if it were new. The safety requirements should be met by scheduling a suitable maintenance period.

Propose the length of time period between maintenances of an equipment t_m, say, so that the safety requirements are satisfied. (Hint: Find a period t_m of time such that the probability that an equipment does not break before the scheduled maintenance is at least 0.9, $P(T > t_m) = 0.9$.) (20 p)

5 Consider traffic accidents in Stockholm, as reported to SRSA, the number of accidents involving trucks with "dangerous goods" signs was 6 during the 4 years between 2000-2003.

(a) Based on the data give an estimate of the risk for at least one accident of a truck having a sign "dangerous goods" during one month, i.e. give a value to $P_t(A)$ where A is an event of at least one accident of truck with dangerous goods and $t = 1$ month. Use a Bayesian approach to compute $P_t(A)$. (Hint: Consider the intensity Λ. Choose an improper Γ-density as a prior: $\Gamma(0,0) = \frac{1}{\lambda}$ to model your prior lack of knowledge.) (13 p)

(b) From other studies one knows that only 50% of trucks having a sign of dangerous goods really contain any dangerous goods. Use the information to give a risk for an accident with a truck really containing dangerous goods. (Assume that the stream of A is independent of event $B =$"Truck with sign really has dangerous goods", $P(B) = 0.5$.) (7 p)

6 Environmentalists studied a small forest for damages of pines due to pollution. They found in total 65 damaged pines in the area of 20000 m^2.

(a) Estimate the intensity of damaged pines in m^{-2}. (3 p)

(b) In order to investigate whether the intensity is constant over the area, the forest has been divided in 25 smaller areas of equal size. Suppose that the intensity is constant and that the locations of damaged trees can be modelled by a Poisson process. Then the number of pines N in the smaller areas are independent Poisson distributed, $N \in \mathrm{Po}(m)$. Give the estimate of m. (5 p)

(c) One has found 1, 4, 5, 11, 2, 2 regions containing 0,1,2,3,4,5 damaged trees. (So, for example there is only 1 region with 0 damaged trees). Use the data to test the hypothesis that the intensity of damaged trees is constant. (12 p)

Good luck!

Solutions: Written examination - 19 August 2009
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1 We search the 50-years snow load s_{50}. It satisfies $F(s_{50}) = 1 - 1/50 = 0.98$, thus $\exp(-(\exp(-\frac{s_{50}-13}{1.5}))) = 0.98$. Solving for s_{50} gives

$$s_{50} = 13 - 1.5 \cdot \ln(-\ln(0.98)) = 18.9.$$

2 Denote the lifetime by the stochastic variable T. The searched probability is $P(T > 15000) = \exp(-\int_0^{15000} \lambda(s)ds) = \exp(-\int_0^{15000} 5 \cdot 10^{-7} + 3 \cdot 10^{-13} \cdot s^2 ds) = \exp(-5 \cdot 10^{-7} \cdot 15000 - 10^{-13} \cdot 15000^3) = 0.71$.

3 From table: $c = 5.73$. Furtermore, $F_X(x_{0.9}) = 1 - e^{-(x/a)^c} = 0.1 \Rightarrow a = \frac{x_{0.9}}{(-\ln 0.9)^{1/c}} = 14.81$. Thus the searched probability is, $P(X < 8) = F_X(8) = 1 - e^{-(8/14.81)^{5.73}} = 0.029$

4 From the hint we have that t_m is equal to 0.9-quantile of T, i.e. it is a solution to the equation

$$\mathsf{P}(T \leq t_m) = 1 - 0.9.$$

Since T is lognormally distributed then one needs to find $m_T = \mathsf{E}[\ln(T)]$ and $\sigma_T^2 = \mathsf{V}[\ln(T)]$. Those are computed using the relations

$$\sigma_T^2 = \ln(1 + \mathsf{R}(T)^2), \quad \mathsf{E}[T] = \exp(m_T + \sigma_T^2/2).$$

Since $\mathsf{R}(T) = 0.25$ we have that $\sigma_T^2 = 0.06$ and $m_T = 1.7618$. Now

$$1 - 0.9 = \mathsf{P}(T \leq t_m) = \Phi\left(\frac{\ln(t_m) - m_T}{\sigma_T}\right)$$

and hence

$$t_m = e^{1.7618 - 1.28\sqrt{0.06}} = 4.25 \quad \text{[moth]}.$$

5 **(a)** The number of accidents in one month is denoted N. The accidents is assumed to follow the Poisson distribution. The intensity of accidents Λ is uncertain and we model it as a random variable. Given the intensity, the probability of at least one accident in the time interval $[0, t]$,

$$P(\text{'at least one accident'}|\Lambda = \lambda) = 1 - P(\text{'No accidents'}|\Lambda = \lambda) = 1 - \exp(-\lambda t).$$

The prior distribution of λ is set to an improper gammafunction, $\Gamma(0,0)$. Using the table of formulæ we obtain the posterior distribution $\Gamma(6,4)$, since 6 is the total number of accidents and 4 the number of years. By the law of total probability the searched probability is obtained, using $t = 1/12$,

$$\begin{aligned} &P(N \geq 1) = 1 - P(N = 0) \\ &= 1 - \int_0^\infty \exp(-\lambda \cdot \frac{1}{12}) \frac{4^6}{\Gamma(4)} \lambda^5 e^{-4\lambda} d\lambda = 1 - \left(\frac{4}{1/12 + 4}\right)^6 = 0.12. \end{aligned}$$

(b) Denote the number of accidents with trucks that really has dangerous goods with M. Similarly to exercise 5 (a), the probability that at least one accident occur during one month is,

$$P(M \geq 1 | \Lambda = \lambda) = 1 - \exp(-\lambda P(B) t),$$

where P(B)=0.5. Again, by the law of total probability we have,

$$\begin{aligned} P(M \geq 1) &= 1 - \int_0^\infty \exp(-P(B)\lambda \cdot \frac{1}{12}) \frac{4^6}{\Gamma(4)} \lambda^5 e^{-4\lambda} d\lambda \\ &= 1 - \left(\frac{4}{P(B)\frac{1}{12} + 4}\right)^6 = 0.06. \end{aligned}$$

6 **(a)** $\lambda^* = \frac{65}{20000} \mathrm{m}^{-2}$.

(b) $m^* = \lambda^* \cdot \text{area} = \lambda^* \cdot \frac{20000}{25} = 2.6$

(c) Hypothesis: $H_0 : N \in \mathrm{Po}(2.6)$. If H_0 is true then $P(N = i) = \exp(-2.6) \cdot \frac{2.6^i}{i!}$. We use the χ^2 test to test the hypothesis. The outcomes are grouped into 5 parts, $N = 0, N = 1, N = 2, N = 3, N \geq 4$, with corresponding probabilities, 0.0743, 0.1931, 0.2510, 0.2176, 0.2640. The number of outcomes in each group is 1, 4, 5, 11, 4.

$$Q = \sum_{i=1}^{5} \frac{(x_i - 25p_i)^2}{25p_i} = 7.5 < \chi^2_{0.05}(5 - 1 - 1).$$

Since the 5%-quantile of the $\chi^2(5 - 1 - 1)$ distribution is 7.81, which is greater than 7.5, we do not reject H_0.