Föreläsning 20/11-13

6.4 Stationary distribution & the limit theorem

π (row matrix) is a stationary distribution for a Markov chain $(X_n; n \geq 0)$ with transition matrix P if π is a discrete probability distribution (that is, $\pi_i \geq 0$ and $\sum_i \pi_i = 1$) and $\pi P = \pi$.

Thm

If $\mu^{(0)} = \pi$ (remember $\mu_i^{(n)} = P(X_n = i)$) then $\mu^{(n)} = \pi$ for all n.
If $\mu^{(m)} = \pi$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\mu^{(n+m)} = \pi$ for all n.

Proof: $\mu^{(n)} = \underbrace{\mu^{(0)}}_{\pi} P^n = \underbrace{(\pi P)}_{\pi} P^{n-1} = \ldots = \pi$ $\blacksquare$

THEOREM

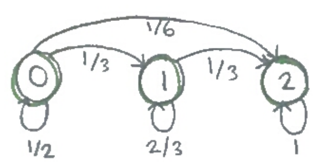
An irreducible chain has a stationary distribution π iff all states are non-null persistent. Then $\pi_i = 1/\mu_i$.
$(\mu_i = E(T_i \mid X_0 = i),\ T_i = \min(n \geq 1 : X_n = i))$

Computer problem

Markov chain $(X_n; n \geq 0)$ with possible values $\{0, 1, 2\}$.
$\mu^{(0)} = (1\ 0\ 0)$

Find $E(T)$ where $T = \min(n \geq 1 : X_n = 2)$.

$$P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$



0, 1 transient

2 persistent

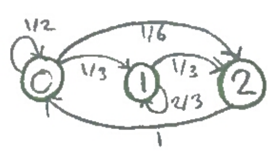
Two analytical solutions of ditto:

1) $E(T) = E(\text{of time until we leave } 0) + \frac{2}{3} E(\text{of time until we leave } 1) =$

$$= \underbrace{\sum_{n=1}^{\infty} 1/2^{n-1}\, 1/2\, n}_{=2} + \frac{2}{3} \underbrace{\sum_{n=1}^{\infty} 2/3^{n-1}\, 1/3\, n}_{=3} = 2 + 3 \cdot 2/3 = 4$$

$$\left(\sum_{n=1} n x^{n-1} = \frac{d}{dx} \sum_{n=1}^{\infty} x^n = \frac{d}{dx} \frac{x}{1-x} \right)$$

2) Consider $P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}$



$E(T) = \mu_2 - 1 = \frac{1}{\pi_2} - 1 = \ldots$

the time it takes to come to 2 from 2.

forts. →

$$(\pi_0\ \pi_1\ \pi_2)\begin{pmatrix}1/2 & 1/3 & 1/6\\ 0 & 2/3 & 1/3\\ 1 & 0 & 0\end{pmatrix} = (\pi_0\ \pi_1\ \pi_2)$$

$$\Leftrightarrow \begin{cases} \frac{1}{2}\pi_0 + \pi_2 = \pi_0 & \pi_2 = 1/2\,\pi_0 \\ \frac{1}{3}\pi_0 + \frac{2}{3}\pi_1 = \pi_1 & \pi_1 = \pi_0 \\ \frac{1}{6}\pi_0 + \frac{1}{3}\pi_1 = \pi_2 & \pi_2 = 1/2\,\pi_0 \\ \pi_0 + \pi_1 + \pi_2 = 1 & \pi_0 + \pi_0 + 1/2\,\pi_0 = 5/2\,\pi_0 = 1 \end{cases}$$

$$\Leftrightarrow \pi_0 = 2/5\ ,\ \pi_1 = 2/5\ ,\ \pi_2 = 1/5$$

THEOREM

For any state j of a Markov chain with period $d(j)=1$.
$(d(j) = GCD\{n \geq 1 : P_{jj}(n) > 0\})$ We have $P_{jj}(n) \longrightarrow 1/\mu_j$ as $n \longrightarrow \infty$ $\underset{=\pi_j}{}$

and $P_{ij}(n) \longrightarrow \frac{f_{ij}}{\mu_j}$ as $n \longrightarrow \infty$ $i \neq j$

$(f_{ij} = P(X_n = j \text{ some } n \geq 1 \mid X_0 = i))$

6.3 Classification of chains

Thm

If $i \leftrightarrow j$ (= intercommunicates) then
a) i and j have same period.
b) i is transient iff j is transient
c) i is null persistent iff j is null persistent.

Proof: There exists $m, n > 0$ such that $\alpha = P_{ij}(m) P_{ji}(n) > 0$
Therefore $P_{ii}(m+r+n) \geq P_{ij}(m) P_{jj}(r) P_{ji}(n) = \alpha P_{jj}(r)$
$P_{jj}(m'+r+n') \geq P_{ji}(m') P_{ii}(r) P_{ij}(n') = \alpha' P_{ii}(r)$

$$\sum_{r=1}^{\infty} P_{ii}(r) = \infty \Leftrightarrow \sum_{r=1}^{\infty} P_{jj}(r) = \infty \ ,$$

i transient iff $\sum_{n=1}^{\infty} P_{ii}(n) < \infty$

$\blacksquare$

i persistent iff $\sum_{n=1}^{\infty} P_{ii}(n) = \infty$

i null pers. iff pers with $\lim_{n\to\infty} P_{ii}(n) = 0$

Definition

A set C of states is called
a) closed if $P_{ij} = 0$ for all $i \in C, j \notin C$
b) irreducible if $i \leftrightarrow j$ $\forall\, ij \in C$.

Thm (decomposition)

The state space S (the set of possible values of the Markov chain) can be decomposed as $S = T \cup C_1 \cup C_2 \ldots$ where T are the transient states and $C_1, C_2, \ldots$ are closed irreducible sets of persistent states.

i transient means $P(X_n = i \text{ some } n \geq 1 \mid X_0 = i) = f_{ii} < 1$

P(that you eventually "escape" from a transient state i and never come back.)

example

$$P = \begin{pmatrix} 1/2 & 1/2 & & & & \\ 1/4 & 3/4 & & & & \\ 1/4 & 1/4 & 1/4 & 1/4 & & \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ & & & & 1/2 & 1/2 \\ & & & & 1/2 & 1/2 \end{pmatrix}$$

$S = \{0, 1, 2, 3, 4, 5\}$

$T = \{2, 3\}$

$C_1 = \{0, 1\}$

$C_2 = \{4, 5\}$

Thm

If S (state space of possible values) is finite then at least one state is persistent and all persistent states are non-null.