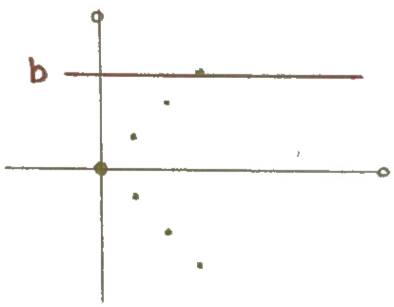
Storgruppsövning 14/11-13

<u>Optional stopping theorem</u>

$(M_n, n \geq 0)$ martingale $\quad (E(M_{n+1} | F_n) = M_n)$
T stopping time $\quad$ (T $\mathbb{N}$-valued r.v. $\{T=n\}$ is F_n-measurable).
Under "technical conditions" $\quad E(M_T) = E(M_0) \quad (= E(M_n))$

example
Showing that "techical conditions" are needed.
Consider random walk $X_n = \sum_{i=1}^{n} Y_i$ where $Y_1, Y_2 \dots$ are IID r.v.'s

$P(Y_i = -1) = P(Y_i = 1) = 1/2$



$T = \min\{n : X_n = b\}$ $\quad$ $b > 0$ is integer

$b = E(X_T) \neq E(X_0) = 0$

5.98
Let $X(t)$ be a Poisson process with rate λ.
Find $P(X(t-d) = k \,|\, X(t) = j)$, $\quad 0 < t-d < t$
Solution:

$P(X(t-d)=k \,|\, X(t)=j) = \dfrac{P(X(t-d)=k, X(t)=j)}{P(X(t)=j)} =$

$= \dfrac{P(X(t-d)=k,\ X(t)-X(t-d)=j-k)}{P(X(t)=j)} = \{\text{independent}\} =$

$= \dfrac{P(P_o(\lambda(t-d))=k)\,P(P_o(\lambda d)=j-k)}{P(P_o(\lambda t)=j)} =$

$= \dfrac{(\cancel{\lambda}(t-d))^k}{k!} \cancel{e^{-\lambda(t-d)}} \dfrac{(\cancel{\lambda} d)^{j-k}}{(j-k)!} \cancel{e^{-\lambda d}} \Big/ \dfrac{(\cancel{\lambda} t)^j}{j!} \cancel{e^{-\lambda t}} =$

$= \binom{j}{k} \left(1-\frac{d}{t}\right)^k \left(\frac{d}{t}\right)^{j-k} = P\left(\mathrm{Bin}(j, 1-d/t) = k\right) \quad \longleftarrow$ Answer

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5.100
Customers arrive at a bank according to Poisson process with rate $\lambda = 6$ (per/hour). These customers are male with probability 2/3 and female with probability 1/3.
Suppose we know that 10 men arrived the first two hours. (*)
How many women would you expect to have arrived in the first two hours.
Solution:
(*) doesn't matter, they are independent Poisson process.
$\Rightarrow \; 12/3 = 4$ $\quad$ look at solved problem 5.58
Answer: 4 women.

5.1 01

Let $X_1, \dots, X_n$ be jointly r.v.'s and let $Y_i = X_i + c_i$ for $i = 1, \dots, n$ where $c_1, \dots, c_n$ are constants. Show that $Y_1, \dots, Y_n$ are also jointly normal r.v.'s.

Solution 1:

Def. 1 $Z_1, \dots, Z_n$ jointly normal r.v.'s $\Leftrightarrow$ $\sum_{i=1}^{n} a_i Z_i$ univariate normal for each choise of constants $a_1, \dots, a_n$.

$X_1, \dots, X_n$ are jointly normal $\Rightarrow$ $\sum_{i=1}^{n} a_i X_i$ normal $\forall a_1, \dots, a_n$.

$$\Rightarrow \sum_{i=1}^{n} a_i (X_i + c_i) \text{ normal } \forall a_1, \dots, a_n = \sum_{i=1}^{n} a_i Y_i \text{ normal}$$

$$\sum_{i=1}^{n} a_i (X_i + c_i) = \sum_{i=1}^{n} a_i X_i + \sum_{i=1}^{n} a_i c_i \qquad \Rightarrow Y_1, \dots, Y_n \text{ jointly normal}$$

$\blacksquare$

Solution 2:

Def. 2 $X_1, \dots, X_n$ jointly normal if $\Psi_{X_1,\dots,X_n}(\omega_1, \dots, \omega_n) = E(e^{j(\omega_1 X_1 + \dots + \omega_n X_n)})$

$$= \exp\left(j\omega^T \mu_X - \tfrac{1}{2}\omega^T k_X \omega\right)$$

where $\omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}$ $\mu_X = \begin{pmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{pmatrix}$ $k_X = \begin{pmatrix} Var(X_1) & Cov(X_1,X_2) & \dots & Cov(X_1,X_n) \\ Var(X_2) & Cov(X_2,X_2) & \dots & Cov(X_2,X_n) \\ & \vdots & & \end{pmatrix}$

$$\Psi_{Y_1,\dots,Y_n}(\omega_1, \dots, \omega_n) = E(e^{j(\omega_1 Y_1 + \dots \omega_n Y_n)}) =$$

$$= E(e^{j(\omega_1 X_1 + \dots + \omega_n X_n)} e^{j(\omega_1 c_1 + \dots + \omega_n c_n)}) =$$

$$= \Psi_{X_1,\dots,X_n}(\omega_1, \dots, \omega_n) e^{j(\omega_1 c_1 + \dots \omega_n c_n)} =$$

$$= \exp\Big(j\omega^T \underbrace{(\mu_X + c)}_{\mu_Y} - \tfrac{1}{2}\omega^T \underbrace{k_X}_{k_Y} \omega\Big)$$

$\blacksquare$

5.1 04

$X_1, X_2 \dots$ are IID r.v.'s, $P(X_i = 3/2) = P(X_i = 1/2) = 1/2$

$M_0 = 1$, $M_n = \prod_{i=1}^{n} X_i = X_1 \cdot X_2 \cdot \dots \cdot X_n$

Show that $\{M_n, n \geq 0\}$ is a martingale.

Solution:

$E(M_{n+1} | F_n) = M_n$?

F_n = info about the history up to time n. = knowledge of $X_1, \dots, X_n$ = = knowledge of $M_1, \dots, M_n$

$$E(M_{n+1} | F_n) = E(\underbrace{X_{n+1}}_{(*)} \underbrace{M_n}_{(+)} | F_n) = \{\text{rule } 4\} = M_n E(X_{n+1} | F_n) = \{\text{rule } 5\} =$$

$$= M_n \underbrace{E(X_{n+1})}_{=1} = M_n$$

$\blacksquare$

(*) - independent of F_n, (+) - F_n-measurable = determined by F_n.

$$\left(E(|M_n|) = E\left(\prod_{i=1}^{n} |X_i|\right) = \prod_{i=1}^{n} \underbrace{E(|X_i|)}_{=1} = 1 \right)$$

5.105

$X_1, X_2, \ldots$ IID r.v.'s $\begin{cases} P(X_i = -1) = q = 1-p \\ P(X_i = 1) = p \end{cases}$

$S_n = \sum_{i=1}^{n} X_i$, $Y_n = \left(\frac{q}{p}\right)^{S_n}$, Show Y_n is martingale

(wrt F_n = info about $X_1, \ldots, X_n$ = info about $Y_1, \ldots, Y_n$ = info about $S_1, \ldots, S_n$)

Solution:

$$E(Y_{n+1} | F_n) = E\Bigg(\underbrace{\left(\frac{q}{p}\right)^{X_{n+1}}}_{\text{indendent of } F_n} \underbrace{Y_n}_{\text{det. by } F_n} \Bigg| F_n\Bigg) = Y_n E\left(\left(\frac{q}{p}\right)^{X_{n+1}}\right) = Y_n \underbrace{\left(\frac{q}{\cancel{p}} \cdot \cancel{p} + \frac{p}{\cancel{q}} \cdot \cancel{q}\right)}_{q+p=1} =$$

$$= Y_n \quad ▨$$

$$\boxed{E(X) = \sum_{\text{all } x} x\, P(x)}$$