Föreläsning 7/11-13

Section 5.8 Martingales

$$E(Y \mid X_1 = x_1, \dots, X_n = x_n) = g(x_1, \dots, x_n) = \int_{-\infty}^{\infty} y \, f_{Y \mid X_1, \dots, X_n}(y \mid x_1, \dots, x_n)\, dy$$

$$E(Y \mid \underbrace{X_1, \dots, X_n}_{\substack{\text{info that we} \\ \text{know } X_1, \dots, X_n \\ \text{denoted } F_n}}) = g(X_1, \dots, X_n) \;\; \text{"} \; E(Y \mid F_n)$$

Properties of conditional expectation:

1) $E(aY_1 + bY_2 \mid F_n) = a\,E(Y_1 \mid F_n) + b\,E(Y_2 \mid F_n)$

2) $E(Y \mid F_n) \geq 0$ if $Y \geq 0$

3) $E(Y \mid F_n) = Y$ if Y is determined by $F_n = (X_1, \dots, X_n)$, Y is F_n-measurable.

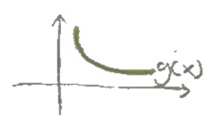
4) (7 i boken) $E(E(Y \mid F_n)) = E(Y) \qquad (= E(g(X_1, \dots, X_n)))$

4) $E(YZ \mid F_n) = Z\,E(Y \mid F_n)$ if Z is determined by $X_1, \dots, X_n$, if Z is F_n-measurable.

6) $E(E(Y \mid F_n) \mid F_m) = E(Y \mid F_m)$ for $m \leq n$

5) $E(Y \mid F_n) = E(Y)$ if Y is independent of F_n

8) $E(g(Y) \mid F_n) \geq g(E(Y \mid F_n))$ for g convex function



$E(g(Y)) \geq g(E(Y))$

example: $g(x) = x^2$, $E(X^2) \geq (E(X))^2$, $Var(X) = E(X^2) - (E(X))^2 \geq 0$.

Definition:

A discrete time random process $(M_n, n \geq 0)$ is a martingale wrt F_n (with respect to)
(where F_n is knowledge about $M_1, \dots, M_n$ if nothing else is said)
if

$E(M_{n+1} \mid F_n) = M_n \quad \forall n$

$E(|M_n|) < \infty \quad \forall n$

$\Longrightarrow E(M_{m+n} \mid F_n) = M_n$ for $m \geq 1$, for $m = 1$ we are done, assume $m \geq 2$

Proof: 6) $\Rightarrow$ $E(M_{m+n} \mid F_n) = E(\underbrace{E(M_{m+n} \mid F_{m+n-1})}_{M_{m+n-1}} \mid F_n) = \dots = E(M_{n+1} \mid F_n) = M_n$

example

$X_n = Y_1 + \dots + Y_n$, $n \geq 0$ where $Y_1, Y_2 \dots$ independent r.v's

F_n = info $Y_1, \dots, Y_n$, is X_n martingale wrt F_n?

$$E(X_{n+1}|F_n) = E(Y_1 + \dots + Y_n + Y_{n+1}|F_n) \overset{1)}{=} E(Y_1 + \dots + Y_n|F_n) + E(Y_{n+1}|F_n) =$$

$$= \{3) + 5)\} = Y_1 + \dots + Y_n + E(Y_{n+1}) = X_n + E(Y_{n+1})$$

Answer: X_n martingale $\Leftrightarrow$ $E(Y_1) = E(Y_2) = \dots = 0$.

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example

$X_n = e^{\sum_{i=1}^{n} Y_i} E(e^{Y_1})^{-n}$ $Y_1, Y_2, \dots$ IID, $n \geq 0$

F_n is knowledge of $Y_1, \dots, Y_n$ martingale?

(—— " —— $X_1, \dots, X_n$)

$$E(X_{n+1}|F_n) = E(e^{\sum_{i=1}^{n+1} Y_i} E(e^{Y_1})^{-(n+1)} | F_n) = E(\underbrace{e^{Y_1 + \dots + Y_n} E(Y_1)^{-n}}_{X_n} e^{Y_{n+1}} E(Y_1)^{-1} | F_n)$$

$$= \{4)\} = X_n E(e^{Y_{n+1}} E(e^{Y_1})^{-1} | F_n) = \{5)\} = E(e^{Y_{n+1}}) E(e^{Y_1})^{-1} X_n =$$

$$= \{IID\} = \underbrace{\frac{E(e^{Y_1})}{E(e^{Y_1})}}_{=1} \cdot X_n = X_n$$

Answer: Yes it is martingale!

Thm $E(M_n) = E(M_0)$ for martingale

Proof $E(M_n) = \{7)\} = E(\underbrace{E(M_n|F_{n-1})}_{M_{n-1}}) = E(M_{n-1}) = \dots = E(M_0)$ ▨

Thm If M_n is martingale wrt F_n and T is a certain $\mathbb{N} = \{0, 1, \dots\}$-valued random variable called stopping time s.t.

$$\left.\begin{array}{l} E(T) < \infty \\ E(|M_T|) < \infty \\ \lim_{n \to \infty} E(|M_n| I_{(T>n)}) = 0 \end{array}\right\} \text{ then } E(M_T) = E(M_0)$$

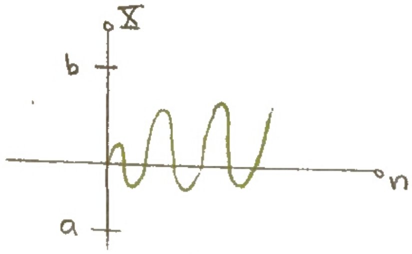
Definition: An $\mathbb{N}$-valued r.v T is stopping time if $\{T = n\}$ is F_n-measurable. That is we can decide if this event happens using info F_n.

example

$X_n = Y_1 + \dots + Y_n$ where $Y_1, Y_2, \dots$ are IID with $\begin{cases} P(Y_i = -1) = 1/2 \\ P(Y_i = 1) = 1/2 \end{cases}$

Take integers $a < 0 < b$, $T = \min\{n : X_n = a \text{ or } X_n = b\}$.

forts. →



$P(X_n$ reaches b before $a) = ?$

Solution: $0 = E(X_0) = E(X_T) = P(X_n$ reaches b before $a) \cdot b +$
$+ (1 - P(X_n$ reaches b before $a)) \cdot a$

$\Rightarrow P(X_n$ reaches b before $a) = \dfrac{-a}{b-a}$

Versions of martingales

submartingales: $E(X_{n+1}|F_n) \geq X_n$

Supmartingales: $E(X_{n+1}|F_n) \leq X_n$

example: X_n martingale, prove $|X_n|$ is submartingale.
$$E(|X_{n+1}||F_n) \geq \{8)\} \geq |\underbrace{E(X_{n+1}|F_n)}_{X_n}| = |X_n|$$

$(M(t), t \geq 0)$ cont. time process, F_t = know. about $(M_s)_{0 \leq s \leq t}$

$E(M_t|F_s) = M_s$ for $s < t$, cont. time mart.

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example: $M(t) = X(t) - \lambda t$ where $X(t)$ is Poisson process with intensity λ.
$$E(M(t)|F_s) = \underbrace{E(M(t)-M(s)|F_s)}_{\substack{\lambda t - \lambda t - (\lambda s - \lambda s) \\ = 0}} + E(M(t)|F_s) = M(s) \qquad (M_s = M(s))$$