$$R_X(s,t) = E(X(s)X(t)) = \left[F_{X(s),X(t)} = F_{X(s-s),X(t-s)}\right] =$$
$$= E(X(0)X(t-s)) = R_X(0, t-s).$$

Föreläsning 31/10-13

Fouriertransform, continuos time

$f: \mathbb{R} \to \mathbb{C}$, $\quad \hat{f}(\omega) = (\mathcal{F}f)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx$, $\quad i = -1$

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\hat{f}(\omega)\,d\omega$$

$f: \mathbb{R}^n \to \mathbb{C}$ $\quad \hat{f}(\omega_1, \dots, \omega_n) = (\mathcal{F}f)(\bar{\omega}) = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} e^{-i(\omega_1 x_1 + \dots + \omega_n x_n)} f(x_1, \dots, x_n)\,dx_1 \dots dx_n$

$$f(x_1, \dots, x_n) = (\mathcal{F}^{-1}\hat{f})(\bar{x}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} \text{———} \ '' \ \text{———}$$

example

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu+i\omega\sigma^2)^2}{2\sigma^2}}}_{I} e^{-\frac{1}{2}\omega^2\sigma^2 - \mu i\omega} \underbrace{dx}_{I}$$

$$= e^{-\frac{1}{2}\omega^2\sigma^2 - \mu i\omega} \quad \text{because } I = 1$$

Properties

$\mathcal{F}(f(x-x_0))(\omega) = e^{-i\omega x_0}(\mathcal{F}f)(\omega)$

$\mathcal{F}(e^{i\omega_0 x} f(x))(\omega) = (\mathcal{F}f)(\omega - \omega_0)$

$\mathcal{F}(f(-x))(\omega) = (\mathcal{F}f)(-\omega)$

$\mathcal{F}(f'(x))(\omega) = i\omega(\mathcal{F}f)(\omega)$

Discrete time

$f: \mathbb{Z} \to \mathbb{C}$, $\quad \hat{f}(\omega) = (\mathcal{F}f)(\omega) = \sum_{k=-\infty}^{\infty} e^{-i\omega k} f(k)$

$$f(k) = (\mathcal{F}^{-1}\hat{f})(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega k}\hat{f}(\omega)\,d\omega$$

Properties

Same as in continuos time, except derivative.

example

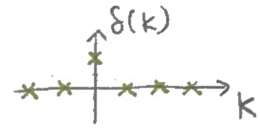
$$f(k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad , k = 0, 1, \dots$$

$$\hat{f}(w) = \sum_{k=-\infty}^{\infty} e^{-ikw} \frac{\lambda^k}{k!} e^{-\lambda} = \underbrace{\sum_{k=-\infty}^{\infty} (e^{-iw}\lambda)^k \frac{1}{k!}}_{\text{Taylor-exp. of } e^{e^{-iw}\lambda}} e^{-\lambda} = e^{(e^{-iw}-1)\lambda}$$

δ-functions

- discrete time δ-function, Kronecker's δ

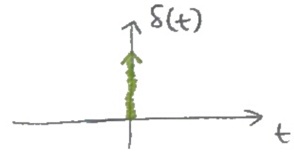
$$\delta(k) = \begin{cases} 0 & k \in \mathbb{Z} \setminus \{0\} \\ 1 & k = 0 \end{cases} \quad , \quad \delta : \mathbb{Z} \to \{0, 1\}.$$



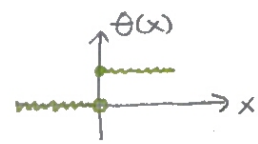
- continuos time δ-function, Dirac's δ

$$\delta : \mathbb{R} \to \{0, \infty\} \quad , \quad \int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$

for f "smooth" with compact support (vanishing at infinity)



- Heavy-side step-function $\quad \theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$



$$\theta'(x) = \delta(x)$$

$$\int_{-\infty}^{\infty} \theta'(x) f(x) dx = \underbrace{\left[\theta x f(x)\right]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} \theta(x) f'(x) dx = 0 - \int_0^{\infty} f'(x) dx = \left[-f(x)\right]_0^{\infty} = f(0)$$

Convolution (= faltning)

$$f, g : \mathbb{R} \to \mathbb{R} \quad , \quad (f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy = \int_{-\infty}^{\infty} g(x-y) f(y) dy = (g * f)(x)$$

$$f, g : \mathbb{Z} \to \mathbb{R} \quad , \quad (f * g)(k) = \sum_{\ell=-\infty}^{\infty} f(k-\ell) g(\ell) = \sum_{\ell=-\infty}^{\infty} g(k-\ell) f(\ell) = (g * f)(k)$$

$$\mathcal{F}(f * g)(w) = (\mathcal{F}f)(w) \cdot (\mathcal{F}g)(w)$$

Proof: $\mathcal{F}(f * g)(w) = \int_{-\infty}^{\infty} e^{-iwx} \int_{-\infty}^{\infty} f(x-y) g(y) dy dx =$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-iw(x-y)} f(x-y)}_{(\mathcal{F}f)(w)} \underbrace{\int_{-\infty}^{\infty} e^{-iwy} g(y) dy}_{(\mathcal{F}g)(w)} dx$$

Properties: $f_X(x) \geq 0$, $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$P(a < X \leq b) = \int_a^b f_X(x)\,dx$, $P(X \in A) = \int_A f_X(x)\,dx$

$f_{X,Y}(x,y) \geq 0$, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$, $P((X,Y)\in A) = \iint_A f_{X,Y}(x,y)$

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$, $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

X and Y independent $\Leftrightarrow$ $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

examples
Gaussian r.v, exponential r.v, uniformly distr. r.v.

expectation $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$, $E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$

Conditional PDF of X given $Y=y$ $\quad f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

$P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx$

$P(X \in A) = \int_{-\infty}^{\infty} P(X \in A \mid Y=y) f_Y(y)\,dy$

$E(X \mid Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$

$E(X) = \int_{-\infty}^{\infty} E(X \mid Y=y) f_Y(y)\,dy$

Discrete random variable X or (X,Y) has finitely or countably infinitely many possible values.

PMF: $P_X(x) = P(X=x)$, $P_{X,Y}(x,y) = P(X=x, Y=y)$

Properties: $P_X(x) \in [0,1]$, $\sum_{\text{all } x} P_X(x) = 1$, $P(X \in A) = \sum_{x \in A} P_X(x)$

$P_{X,Y}(x,y) \in [0,1]$, $\sum\sum_{\text{all } x,y} P_{X,Y}(x,y) = 1$, $P((X,Y)\in A) = \sum\sum_{(x,y)\in A} P_{X,Y}(x,y)$

$P_X(x) = \sum_{\text{all } y} P_{X,Y}(x,y)$, $P_Y(y) = \sum_{\text{all } x} P_{X,Y}(x,y)$

X and Y independent $\Leftrightarrow$ $P_{X,Y}(x,y) = P_X(x) P_Y(y)$

$Z = g(X,Y)$, $f_Z(z) = \frac{d}{dz} P(g(X,Y) \le z) =$
$= \frac{d}{dz} \iint_{g(x,y)\le z} f_{X,Y}(x,y)\,dx\,dy$

Föreläsning 31/10-13

Random process is a collection $(X(t);\, t \in T)$ of r.v.'s indexed by time $t \in T$.

$\mu_X(t) = E(X(t))$ mean function

$R_X(s,t) = E(X(s)X(t))$ correlation function

$K_X(s,t) = Cov(X(s), X(t))$ covariance function

Stationarity

Strict stationarity $F_{X(t_1),\dots,X(t_n)}(x_1,\dots,x_n) = F_{X(t_1+h),\dots,X(t_n+h)}(x_1,\dots,x_n)$

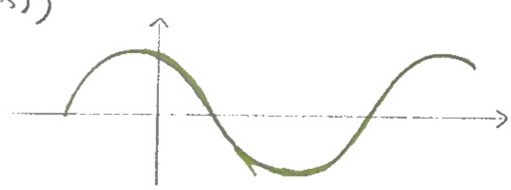
WSS weak/wide stationarity $\mu_X(t) = C$ does not depend on $t = \mu_X$

$R_X(s,t) =$ function of $t-s$ only $= R_X(t-s)$

example

Cosine process $X(t) = A\cos(\omega t) + B\sin(\omega t)$
A, B independent $N(0,1)$ r.v.'s , $\omega \in \mathbb{R}$ constant
$\mu_X(t) = 0$, $R_X(s,t) = \cos(\omega(t-s))$

normal process $\sum_{i=1}^{n} a_i X(t_i) =$
$= \left(\sum_{i=1}^{n} a_i \cos(\omega t_i)\right) A + \left(\sum_{i=1}^{n} a_i \sin(\omega t_i)\right) B$



WSS

Normal processes = Gaussian processes

Definition $(X(t), t \in T)$ is normal process if each linear combination $\sum_{i=1}^{n} a_i X(t_i)$ is (1-dimensionally) normal distributed.

Characteristic functions (continuos r.v's)

$\Psi_X(\omega) = E(e^{i\omega X}) = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x)\,dx$

$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \Psi_X(\omega)\,d\omega$

$\Psi_{X_1,\dots,X_n}(\omega_1,\dots,\omega_n) = E\left(e^{i(\omega_1 X_1 + \dots + \omega_n X_n)}\right) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{i(\omega_1 x_1 + \dots + \omega_n x_n)} f_{X_1,\dots,X_n}(x_1,\dots,x_n)\,dx_1\dots dx_n$

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n)=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i(w_1x_1+\dots+w_nx_n)}\,\Psi_{X_1,\dots,X_n}(w_1,\dots,w_n)\,\overbrace{dw_1\dots dw_n}$$

$$\Psi_X^{(n)}(0)=\frac{d}{dw^n}\Psi_X(w)\Big|_{w=0}=E\left((iX)^n e^{iwX}\right)\Big|_{w=0}=i^n E(X^n)$$

$$\Psi^{(k_1\dots k_n)}_{X_1,\dots,X_n}(0,\dots,0)=i^{k_1+\dots+k_n}E\left(X_1^{k_1}\dots X_n^{k_n}\right)$$

example

X, $N(\mu,\sigma^2)$, $f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\Psi_X(w)=e^{iw\mu-\frac{1}{2}w^2\sigma^2}$, $\Psi_X'(0)=(i\mu-w\sigma^2)e^{\cdots}\Big|_{w=0}=i\mu\Rightarrow E(X)=\mu.$

$\Psi_X''(0)=\dots\Rightarrow E(X^2)=\mu^2+\sigma^2$

$$\Psi_{X(t_1),\dots,X(t_n)}(w_1,\dots,w_n)=E\Big(e^{i\underbrace{(w_1X(t_1)+\dots+w_nX(t_n))}_{N(\mu,\sigma^2)}}\Big)=$$

$$=e^{iw\mu-\frac{1}{2}w^2\sigma^2}\quad\text{where}\quad\begin{cases}\mu=E\left(\sum_{i=1}^n w_iX(t_i)\right)=\sum_{i=1}^n w_i\mu_X(t_i)\\ \sigma^2=Var\left(\sum_{i=1}^n w_iX(t_i)\right)=\sum_{i=1}^n\sum_{j=1}^n w_iw_jK_X(t_i,t_j)\end{cases}$$

$$\sigma^2 = Cov\left(\sum_{i=1}^n w_iX(t_i),\sum_{j=1}^n w_jX(t_j)\right)$$

Corollary (= följdsats)

All probabilities (all multi-dim PDF's etc.) for normal processes are determined by μ_X and K_X.

Corollary

Normal (r.v's) process values are independent $\Leftrightarrow$ uncorrelated ($cov=0$)

Corollary

A WSS normal process is strictly stationary.

Bivariate normal random variable (X,Y) with parameters $\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2$ and $\rho=$ correlation coefficient $\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2-2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)}{2(1-\rho^2)}\right)$$

n-variate r.v

$(X_1,\dots,X_n)$, $\mu=\begin{pmatrix}\mu(X_1)\\ \vdots\\ \mu(X_n)\end{pmatrix}$ $\quad K=\begin{pmatrix}Var(X_1) & Cov(X_1,X_2) & \dots & Cov(X_1,X_n)\\ & & & \vdots\\ & & & Cov(X_n,X_n)\end{pmatrix}$

$K_{ij}=Cov(X_i,X_j)$

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(K)}} \cdot \exp\left(-\frac{(x-\mu)^T K^{-1}(x-\mu)}{2}\right),$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Markov process

$(X)_{n=0}^{\infty} = (X_n, n \in \mathbb{N})$

Markov property: $P(\underset{\substack{\uparrow \\ \text{future}}}{X_{n+1} = i_{n+1}} \mid \underset{\substack{\uparrow \\ \text{now}}}{X_n = i_n}, \underbrace{\dots X_0 = i_0}_{\text{earlier history}}) = P(X_{n+1} = i_{n+1} \mid X_n = i_n)$

example

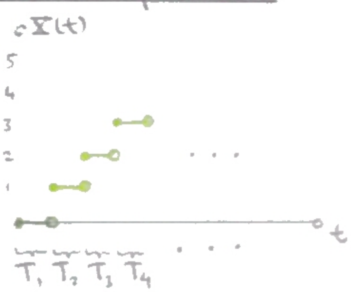
Random walk

$X_n = \sum_{i=1}^{n} Y_i$ where $Y_1, Y_2, \dots$ independent identically distributed

$P(Y_i = 1) = p$, $P(Y_i = -1) = 1 - p$

$P(X_{n+1} = i_{n+1} \mid X_n = i_n, \dots, X_0 = i_0) = P(Y_{n+1} + X_n = i_{n+1} \mid X_n = i_n, \dots, \cancel{X_0} = i_0)$

Poisson process



$\iff$

$X(t)$ is $\mathbb{N}$-valued process s.t
$X(0) = 0$
$X(t) - X(s)$ $Po(\lambda(t-s))$-distributed, $s \le t$
$X(t) - X(s)$ is independent of all earlier process values $X(r)$, $r \in [0, s]$

$T_1, T_2, \dots$ independent identically exp-distributed with param. λ (mean $1/\lambda$)

$$\mu_X(t) = E(X(t)) = \sum_{k=0}^{\infty} k\, P_{X(t)}(k) =$$
$$= \sum_{k=0}^{\infty} k \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \lambda t \underbrace{\sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!}}_{\text{Taylor } e^{\lambda t}} e^{-\lambda t} = \lambda t$$

$$\underset{s \le t}{K_X(s,t)} = Cov(X(s), X(t)) = Cov(X(s), X(s)) + \underbrace{Cov(X(s), X(t) - X(s))}_{=0} =$$
$$= Var(X(s)) = \lambda s \; (= \lambda \min\{s,t\} \text{ in general})$$

Stationary independent increment process

$(X(t), t \ge 0)$ is stationary independent increment process

$X(t) - X(s)$ has a probability distribution that depends on $t - s$ only (stationary increments)

$X(t) - X(s)$ is independent of $(X(r))$, $r \in [0, s]$ for $s \le t$ (independent increments)