Föreläsning 30/10-13

Random processes (ch. 5 in Hsu's book)

Random variable $X = X(\zeta)$ a function of the outcome $\zeta \in S$ of a random experiment with possible outcomes in the sample space S.

CDF (cummulative Distribution Function)
$F_X(x) = P(X \le x) = P(\zeta \in S : X(\zeta) \le x)$

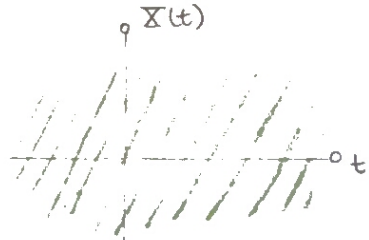
Continuos r.v **PDF** Probability Density Function
$f_X(x) = \frac{d}{dx} F_X(x)$

Discrete r.v **PMF** Probability Mess Function
$P_X(x) = P(X = x)$ for possible values x of X.

Random process is a collection of r.v $X(t) = X(t, \zeta)$ indexed by a parameter (usually thought of as time) $t \in T$. $(X(t), t \in T)$

example
White noise

Completely independent process.

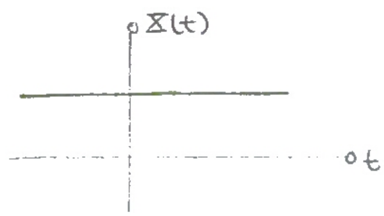


For each $t \in \mathbb{R}$ you get an independent $N(0,1)$-distributed process value $X(t)$

$F_{X(t)}(x) = P(N(0,1) \le x) = \Phi(x)$
$\mu_X(t) = 0, \; R_X(s,t) = \begin{cases} 0 & s \ne t \\ 1 & s = t \end{cases}$

example

Completely dependent process

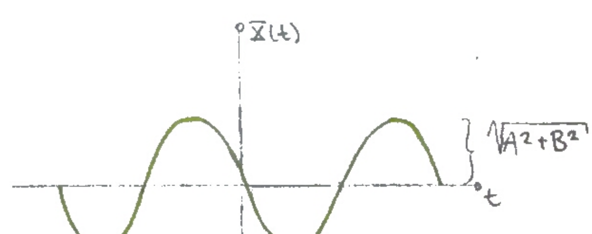


$X(t) = Y$ for all $t \in \mathbb{R}$ where Y is one single $N(0,1)$-distributed r.v - the same Y for all t.

$F_{X(t)}(x) = P(N(0,1) \le x) = \Phi(x)$
$\mu_X(t) = 0, \; R_X(s,t) = 1$ all s,t

example (cosine process)

Let A and B be independent $N(0,1)$-distributed r.v's and $w \in \mathbb{R}$ is a constant

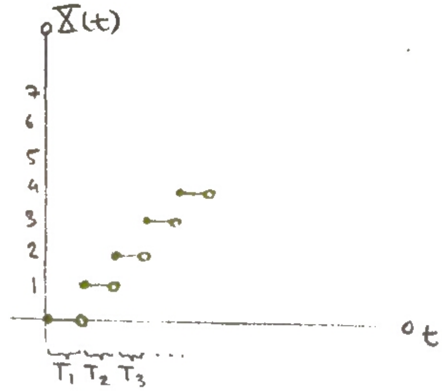


$X(t) = A\cos(wt) + B\sin(wt)$

$\mu_X(t) = 0, \; R_X(s,t) = E((A\cos(ws) + B\sin(ws))(A\cos(wt) + B\sin(wt))) =$
$= \underbrace{E(B^2)}_{=1}\sin(ws)\sin(wt) + \underbrace{E(A^2)}_{=1}\cos(ws)\cos(wt) + \underbrace{E(AB)}_{=0}\text{cossin} + \underbrace{E(BA)}_{=0}\text{cossin} =$
$= \cos(w(t-s))$

example

Poisson process



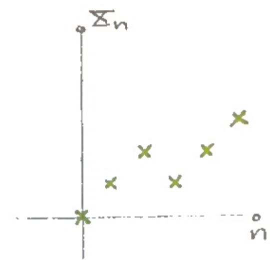
where $T_1, T_2, T_3 \dots$ are independent exponentially distibuted r.v's.

CDF $F_{X(t)}(x) = P(X(t) \le x)$ for each $t \in T$
insufficient info to determine probabilistic behaviour of process

n-dimensional CDF $F_{X(t_1),\dots,X(t_n)}(x_1,\dots,x_n) = P(X(t_1) \le x_1, \dots, X(t_n) \le x_n)$
sufficient to determine probabilistic behaviour of process.

example
Simple random walk
$Y_1, Y_2, \dots$ independent identically distributed r.v's
s.t $P(Y_i = 1) = p$, $P(Y_i = -1) = 1-p$
$X_n = \sum_{i=1}^{n} Y_i$

$$\mu_X(n) = E(X_n) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) =$$
$$= nE(Y_1) = n(1\cdot p + (-1)\cdot(1-p)) = n(2p-1)$$



Mean function $\mu_X(t) = E(X(t))$

Correlation function $R_X(s,t) = E(X(s)X(t))$

Covariance function $K_X(s,t) = Cov(X(s), X(t)) =$
$= E((X(s) - \mu_X(s))(X(t) - \mu_X(t))) = E(X(s)X(t)) - \mu_X(s)\mu_X(t) =$
$= R_X(s,t) - \mu_X(s)\mu_X(t)$

wide
Strict sense and weak sense stationarity
$X(t)$ is strict sense stationary if $F_{X(t_1+h),\dots,X(t_n+h)}(x_1,\dots,x_n) =$
$= F_{X(t_1),\dots,X(t_n)}(x_1,\dots,x_n)$

$X(t)$ is weak/wide sense stationary (WSS) if
$\mu_X(t)$ does not depend on t and $R_X(s,t)$ only depends on the distance between s and t.

Result: strict stationary $\Longrightarrow$ WSS , when μ_X and R_X are well-defined

Proof: Assume $X(t)$ strict stationary,
$\mu_{X(s)} = E(X(s)) = \left[F_{X(s)}(x) = F_{X(t)}(x)\right] = E(X(t)) = \mu_{X(t)}$ and

$$R_X(s,t) = E(X(s)X(t)) = \left[F_{X(s),X(t)} = F_{X(s-s),X(t-s)}\right] =$$
$$= E(X(0)X(t-s)) = R_X(0, t-s).$$