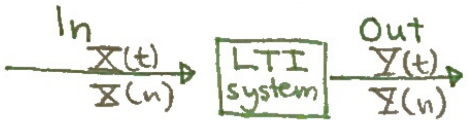
Föreläsning 4/12-13

Analysis and processing of random variables (ch. 6 Hsu)

WSS random process $\{X(t);\, t \geq 0\}$ or $\{X(n);\, n \in \mathbb{N}\}$
sending X through LTI (Linear Time Invariant) system (filter).



(t-continous
n-discrete)

Let $h(t)/h(n)$ be the impuls response from LTI.

$h(t) = (T\delta)(t)$ $\quad \delta$ = Dirac delta

$h(n) = (T\delta)(n)$ $\quad \delta$ = Kronecker delta

$$Y(t) = (TX)(t) = T \int_{-\infty}^{\infty} \delta(t-u) X(u)\,du = \{\text{linearity}\} = \int_{-\infty}^{\infty} T(\delta(t-u)) X(u)\,du =$$

$$= \{\text{time-invariance } T(f(t-u)) = (Tf)(t-u)\} = \int_{-\infty}^{\infty} (T\delta)(t-u) X(u)\,du =$$

$$= \{(T\delta)(t-u) = h(t-u)\} = \int_{-\infty}^{\infty} h(t-u) X(u)\,du = (h * X)(t)$$

$$Y(n) = (TX)(n) = T \sum_{k=-\infty}^{\infty} \delta(n-k) X(k) = \{\text{linearity}\} = \sum_{k=-\infty}^{\infty} T(\delta(n-k)) X(k) =$$

$$= \{h \text{ impuls response, time-invariance}\} = \sum_{k=-\infty}^{\infty} h(n-k) X(k) = (h * X)(n)$$

WSS process — autocorrelation fcn $R_X(s,t) = E(X(t)X(s)) = R_X(t-s)$
or $R_X(t, t+T) = E(X(t)X(t+T)) = R_X(T)$
\ mean function $\mu_X = E(X(t))$ does not depend on t.

mean

$$\mu_Y = E(Y(t)) = E\left(\int_{-\infty}^{\infty} h(t-u) X(u)\,du\right) = \{\text{linearity}\} = \int_{-\infty}^{\infty} h(t-u) \underbrace{E(X(u))}_{\mu_X} du =$$

$$= \left(\int_{-\infty}^{\infty} h(u)\,du\right) \mu_X$$

$$\mu_Y = E(Y(n)) = \quad \ldots \text{ p.s.s } \ldots \quad = \left(\sum_{k=-\infty}^{\infty} h(k)\right) \mu_X$$

cross-correlation

$$R_{XY}(T) = E(X(t)Y(t+T)) = E\left(X(t) \int_{-\infty}^{\infty} h(t+T-u) X(u)\,du\right) =$$

$$= \int_{-\infty}^{\infty} h(t+T-u) \underbrace{E(X(t)X(u))}_{R_X(u-t)} du = \left\{\begin{array}{l}\text{change of variable} \\ \text{in integral, } v = u-t\end{array}\right\} =$$

$$= \int_{-\infty}^{\infty} h(T-v) R_X(v)\,dv = (h * R_X)(T) = (*)$$

$$R_{XY}(T) = E(X(n)Y(n+T)) = \ldots \text{ p.s.s } \ldots = \sum_{k=-\infty}^{\infty} h(T-k) R_X(k) = (*)$$

$$R_Y(T) = E(Y(t)Y(t+T)) = E\left(\int_{-\infty}^{\infty} h(t-u) X(u)\,du \int_{-\infty}^{\infty} h(t+T-v) X(v)\,dv\right) =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-u) h(t+T-v) \underbrace{E(X(u)X(v))}_{R_X(v-u)} du\,dv =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-u)h(t+\tau-v)R_X(v-u)\,du\,dv =$$

$$= \left\{ \int_{v=-\infty}^{\infty} h(t+\tau-u-(v-u))R_X(v-u)\,dv = (h * R_X)(t+\tau-u) \right\} =$$

$$= \int_{u=-\infty}^{\infty} h(t-u)\,(h*R_X)(t+\tau-u)\,du = \{-\hat{u} = t-u\} =$$

$$= \int_{\hat{u}=-\infty}^{\infty} h(-\hat{u})\,(h*R_X)(\tau-\hat{u})\,d\hat{u} = \big(h(-\cdot) * h * R_X\big)(\tau)$$

Fourier transform

$$(Fg)(\omega) = \hat{g}(\omega) = \begin{cases} \int_{-\infty}^{\infty} e^{-i\omega t} g(t)\,dt & \text{— continuous time} \\ \sum_{k=-\infty}^{\infty} e^{-i\omega k} g(k) & \text{— discrete time} \end{cases}$$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{g}(\omega)\,d\omega$$

$$g(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega k}\hat{g}(\omega)\,d\omega$$

$$F(h*g) = (Fh)(Fg)$$

PSD = Power Spectral Density of WSS process $X(t)$/$X(n)$

$$S_X(\omega) = (FR_X)(\omega) = \begin{cases} \int_{-\infty}^{\infty} e^{-i\omega\tau} R_X(\tau)\,d\tau & R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} S_X(\omega)\,d\omega \\ \sum_{k=-\infty}^{\infty} e^{-i\omega\tau} R_X(k) & R_X(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega k} S_X(\omega)\,d\omega \end{cases}$$

Frequency response Transfer function:

$$H(\omega) = (Fh)(\omega) = \begin{cases} \int_{-\infty}^{\infty} e^{-i\omega t} h(t)\,dt \\ \sum_{k=-\infty}^{\infty} e^{-i\omega k} h(k) \end{cases}$$

Cross PSD:

$$S_{XY}(\omega) = (FR_{XY})(\omega) = \begin{cases} \int_{-\infty}^{\infty} e^{-i\omega\tau} R_{XY}(\tau)\,d\tau \\ \sum_{k=-\infty}^{\infty} e^{-i\omega k} R_{XY}(k) \end{cases}$$

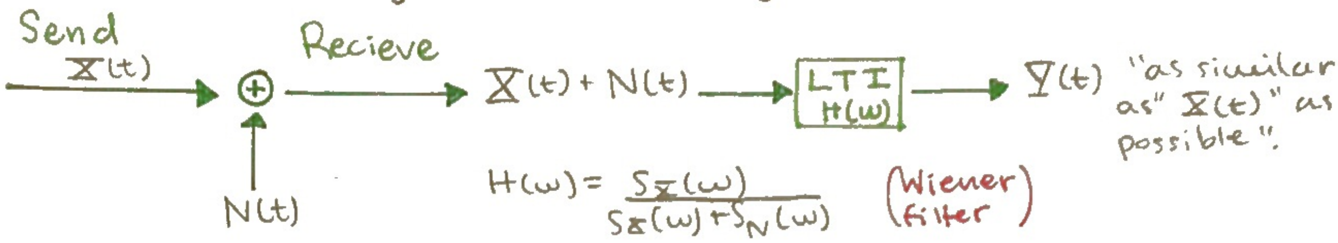
$$S_{XY}(\omega) = (FR_{XY})(\omega) = F(h * R_X)(\omega) = (Fh)(FR_X)(\omega) = H(\omega)S_X(\omega)$$

$$S_Y(\omega) = (FR_Y)(\omega) = F(h * h(-) * R_X)(\omega) = \overbrace{(Fh)}^{H}(Fh(-))\overbrace{(FR_X)(\omega)}^{S_X} =$$

$$= H(\omega)\overline{H}(\omega) S_X(\omega) = |H(\omega)|^2 S_X(\omega)$$

example

Transmission of signal $X(t)$ on noisy channel with noise $N(t)$



$$H(\omega) = \frac{S_X(\omega)}{S_X(\omega) + S_N(\omega)}$$ (Wiener filter)

White noise 6.4 Hsu

White noise is WSS process $W(t)/W(n)$ with $S_W(\omega) = \sigma^2 = \text{constant}$, gives that:

$$\begin{cases} R_X(\tau) = \sigma^2 \delta(\tau) & \text{in continous time} \\ R_X(k) = \sigma^2 \delta(k) & \text{in discrete time} \end{cases}$$

(

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{-|t|} dt = 2/(1+\omega^2)$$

Fourier transforms you should recognise

$$\int_{-\infty}^{\infty} e^{-i\omega t} \frac{1}{1+t^2} dt = \pi e^{-|\omega|}$$

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