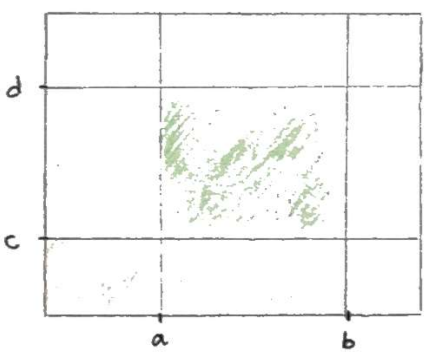
Föreläsning 1 — CRASH COURSE

TWO - (high) DIM. PROB. THEORY

* Two dim random variable $(X ; Y) : S \to \mathbb{R}$
* Cumulative distr. fct (CDF) $F_{XY}(x,y) = P(X \le x, Y \le y)$

properties:

- $0 \le F_{XY}(x,y) \le 1$
- $F_{XY}(x,y)$ is increasing in each arg.
- $F_{XY}(x,\infty) = F_X(x)$, $F_{XY}(\infty, y) = F_Y(y)$
- $F_{XY}(x,-\infty) = F_{XY}(-\infty,y) = 0$
- $P(a < X \le b, c < Y \le d) = F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,c) + F_{XY}(a,c)$



$\forall x,y \in \mathbb{R}$

* Two rand. var. X & Y are indep if $P(X \le x, Y \le y) = P(X \le x)\, P(Y \le y)$

property: X, Y indep $\Leftrightarrow$ $F_{XY}(x,y) = F_X(x)\, F_Y(y)$ $\forall x,y$

* Continuous rand. var. (X,Y) has uncountably infinitly many possible values

* Cont. r.v has prob. density fct PDF $f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$

properties

- $f_{XY}(x,y) \ge 0$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$
- $F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\,du\,dv$
- $P((X,Y) \in A) = \iint_{(x,y)\in A} f_{XY}(x,y)\,dx\,dy$
-

• $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$, $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

* X & Y are indep $\Leftrightarrow$ $f_{XY}(x,y) = f_X(x) f_Y(y)$

IMPORTANT ONE DIM CONT. PROB. DISTR.

* Gaussian / normal $\quad f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$

* Exponential $\quad f(x) = \lambda e^{-\lambda x} \quad x \in [0,\infty)$

* Uniform $\quad f(x) = \frac{1}{b-a} \quad x \in [a,b]$

⋮

TWO DIM (again) CONT.

* Expectation $E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\,dx\,dy$
  for fct $g: \mathbb{R}^2 \to \mathbb{R}$

* Conditional PDF of X given $Y=y$
  $f_{X|Y}(x|y) := \frac{f_{XY}(x,y)}{f_Y(y)}$

* Def: $P(X \in A \mid Y=y) = \int_{x\in A} f_{X|Y}(x|y)\,dx \quad A \subseteq \mathbb{R}$
  $E(X|Y=y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx$

properties:
• $P(X \in A) = \int_{-\infty}^{\infty} P(X\in A \mid Y=y) f_Y(y)\,dy$ Tot prob
• $E(X) = \int_{-\infty}^{\infty} E(X|Y=y) f_Y(y)\,dy$ Tot exp val

proof first: $\int_{-\infty}^{\infty} P(X\in A|Y=y) f_Y(y)\,dy = \int_{-\infty}^{\infty}\int_{x\in A} f_{X|Y}(x,y)\,dx\, f_Y(y)\,dy$
$= \iint_{(x,y)\in A\times(-\infty,\infty)} f_{XY}(x,y)\,dx\,dy = P(X\in A, Y\in(-\infty,\infty)) = P(A)$

TWO DIM DISCRETE RAND VAR

* Has finitely or countably many possible values (eg. $\mathbb{Z} \times \mathbb{Z}$)

* Probability mass function (PMF)

$$P_{XY}(x,y) = P(X=x, Y=y)$$

properties:
- $P_{XY}(x,y) \geq 0$
- $\sum_{x,y} P_{XY}(x,y) = 1$
- $F_{XY}(x,y) = \sum_{u \leq x} \sum_{v \leq y} P_{XY}(u,v)$
- $P(XY \in A) = \sum_{(x,y) \in A} P_{XY}(x,y)$ for $A \subseteq \mathbb{R}^2$
- $P_X(x) = \sum_y P_{XY}(x,y)$ $\quad P_Y(y) = \sum_x P_{XY}(x,y)$

* X & Y indep $\iff$ $P_{XY}(x,y) = P_X(x)\, P_Y(y)$

IMPORTANT ONE DIM DISC PROB DISTR

* Binomial $\binom{n}{k} p^k (1-p)^{n-k}$ pmf

* Bernoulli $pX(1) = p$ $\quad pX(0) = 1-p$

* Poisson $\frac{\lambda^k e^{-\lambda}}{k!}$

* Geometric/waiting time $(1-p)^k p$

TWO DIM ... (again) DISC

* $E(g(X,Y)) = \sum_x \sum_y g(x,y)\, P_{XY}(x,y)$ for $g: \mathbb{R}^2 \to \mathbb{R}$

* Conditional PMF of X given $Y=y$: $P_{X|Y}(x|y) = P(X=x \mid Y=y)$

$$= \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{P_{XY}(x,y)}{P_Y(y)}$$

Facts $P(X \in A \mid Y=y) = \sum_{x \in A} P_{X|Y}(x|y)$

$E(X \mid Y=y) = \sum_{x} x \, P_{X|Y}(x|y)$

properties
- $P(X \in A) = \sum_{y} P(X \in A \mid Y=y) \, P_Y(y)$
- $E(X) = \sum_{y} E(X \mid Y=y) \, P_Y(y)$

* Linearity of the mean : $E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$

* Variance : $Var(X) = \sigma_x^2 = E((X - E(X))^2) = E(X^2) - \overbrace{E(X)}^{\mu}{}^2$

* Covariance : $Cov(X,Y) = E((X-\mu_x)(Y-\mu_y)) = E(X,Y) - \mu_x \mu_Y$

* Def : X & Y are uncorrelated if $Cov(X,Y) = 0$
$(\Leftrightarrow E(X,Y) = E(X)E(Y))$

* Simple fact : X, Y indep $\Rightarrow$ X, Y uncor.

proof (cont) :

$$E(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{XY}(x,y) \, dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_X(x) f_Y(y) \, dx\,dy$$

$$= \int x f_X(x) \, dx \int y f_Y(y) \, dy = E(X)\,E(Y)$$

* Bilinearity of covariance : $Cov\left(\sum_{i=1}^{m} a_i X_i \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \, Cov(X_i Y_j)$

* $Var(X) = Cov(X,X)$

Consequence • $Var\left(\sum_{i=1}^{m} a_i X_i\right) = Cov\left(\sum_{i=1}^{m} a_i X_i \sum_{j=1}^{m} a_j X_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \, Cov(X_i, X_j)$

$=$ [if X_i's uncor.] $= \sum_{i=1}^{m} a_i^2 \, Var(X_i)$

* Characteristic fct (CHF) of rand var X : $\Psi_X(\omega) = E(e^{j\omega X})$

X cont : $\Psi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x) \, dx$ fourier transform

$\Rightarrow f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_X(\omega) \, e^{-j\omega x} \, d\omega$ inverse transform

* $\Psi_{X_1,\dots,X_n}(\omega_1,\dots,\omega_n) = \int\cdots\int e^{j(\omega_1 x_1 + \dots \omega_n x_n)} f_{X_1\dots X_n}(x_1\dots x_n)$

multi dim

$$f_{\bar{X}}(\bar{x}) = \frac{1}{(2\pi)^n}\int\cdots\int e^{-j(\bar{\omega},\bar{x})}\,\Psi_{\bar{X}}(\bar{\omega})\;d\bar{\omega}$$

Vector form inverse

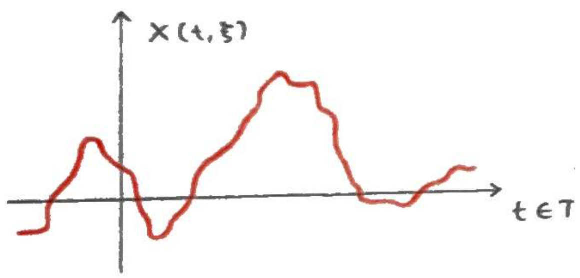
Förelāsning 2 CH 5

Today: 5.1 - 5.4

A random (stochastic) process is a family $\{X(t)\}_{t\in T} = \{X(t,\xi)\}_{t\in T}$ of random variables indexed by time $t \in T$ where $\xi \in S$ is the outcome of a random experiment

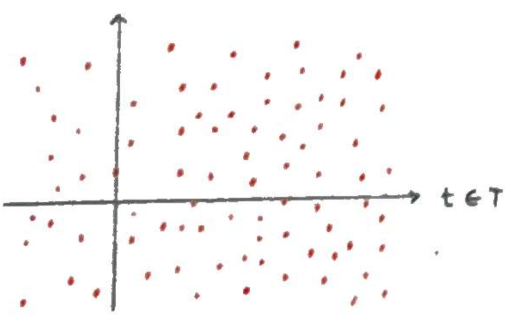
The time parameter set T is either discrete (e.g $\mathbb{N}$, $\mathbb{Z}$, $\{0,1,\dots,n\}$) or continuous (eg. $\mathbb{R}^+$, $\mathbb{R}$, $[a,b]$)

When you have done your rand. exp. & recieved its outcome $j \in S$ you can print the random process



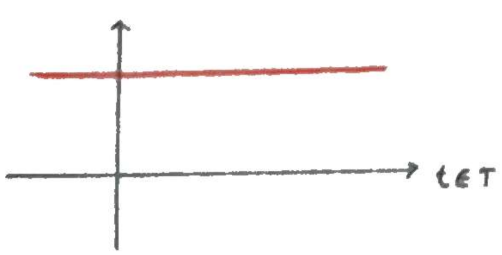
Sample path of the rand. var for some ξ

Ex 1: Totally independent process



each process-value is a $N(0,1)$ rand. var. indep. of

Ex 2: Totally dependent process



$X_2(t) = g \quad \forall t \in T$
where g is a single $N(0,1)$ rand. var.

$g \sim N(0,1)$

$$F_{X_1(t)}(x) = P(X_1(t) \le x) = P(N(0,1) \le x) = \Phi(x)$$
$$F_{X_2(t)}(x) = P(X_1(t) \le x) = P(N(0,1) \le x) = \Phi(x)$$

What is really required to know all probabilistic info. about the random process are the finite dim distributions

$$F_{X(t_1)\dots X(t_n)}(x_1 \dots x_n) = P(X(t_1) \le x_1 \dots X(t_n) \le x_n) \quad \forall\ t_1, \dots, t_n \in T,\ x_1, \dots, x_n \in \mathbb{R},\ n \in \mathbb{N}$$

n-dim probability

The mean fct : $\mu_X(t) = E(X(t))$

The autocorrelation fct : $R_X(s,t) = E(X(s)\, X(t))$

Cross correlation fct : $R_{XY}(s,t) = E(X(s)\, Y(t))$

Autocovariance fct : $K_X(s,t) = Cov(X(s)\, X(t))$

Cross covariance fct : $K_{XY}(s,t) = Cov(X(s)\, Y(t))$

$$R_X(s,t) = E(X(s)\, X(t)) = Cov(X(s)\, X(t)) + E(X(s))\, E(X(t))$$
$$= K_X(s,t) + \mu_X(s)\, \mu_X(t)$$

$$R_{XY}(s,t) = \dots = K_{XY}(s,t) + \mu_X(s)\, \mu_Y(t)$$

$X(t)$ is strictly stationary if $P(X(t_1+h) \le x_1, \dots, X(t_n+h) \le x_n)$
$= P(X(t_1) \le x_1, \dots, X(t_n) \le x_n)$
$\forall\ t_1+h, \dots, t_n+h \in T,\ x_1, \dots, x_n \in \mathbb{R},\ h \in \mathbb{R},\ n \in \mathbb{N}$

ie. time translation invariance of the finite dim. distributions

$X(t)$ is wide / weak sense stationary (wss) if $\mu_X(t) = \mu_X$ &
$R_X(t, t+\tau)$ does not depend on t

Thm

Strict Stationarity $\Rightarrow$ WSS

proof: $E(X(t))$ is determined by $F_{X(t)}(x)$ which does not depend on t

$E(X(t)X(t+T))$ is determined by $F_{X(t)X(t+T)}(x,y)$ —"—

Ex 1 cont.

$$P(X_1(t_1+h) \le x_1, \dots, X_1(t_n+h) \le x_n) = P(X_1(t_1+h) \le x_1) \cdots P(X_1(t_n+h) \le x_n)$$

$= P(N(0,1) \le x_1) \cdots P(N(0,1) \le x_n)$ for $t_1+h, \dots, t_n+h \in \mathbb{R}$ all different

which does not depend on h, So that $X_1(t)$ strictly stationary

Ex 2. cont.

$$P(X_2(t_1+h) \le x_1, \dots, X_2(t_n+h) \le x_n) = P(\eta \le x_1, \dots, \eta \le x_n)$$

$= P(N(0,1) \le \min(x_1, \dots, x_n))$ for $t_1+h, \dots, t_n+h \in \mathbb{R}$

which does not depend on h, So that $X_2(t)$ is strictly Stationary

Ex 3. $X_3(t) = U\cos wt + V\sin wt$

where U & V are uncorrelated zero mean random Variables with common variance σ^2, and $w \in \mathbb{R}$ is constant.

$$\mu_{X_3}(t) = E(X_3(t)) = \cos wt\, E(U) + \sin wt\, E(V) = 0$$

by linearity of the mean

$$R_{X_3}(t, t+T) = E\big((U\cos wt + V\sin wt)(U\cos w(t+T) + V\sin w(t+T))\big)$$

$$= \overset{\sigma^2}{E(U^2)} \cos wt \cos w(t+T) + \overset{\sigma^2}{E(V^2)} \sin wt \sin w(t+T) + \overset{0}{E(UV)}(\cos wt \sin w(t+T) + \sin wt \cos w(t+T))$$

$$= \sigma^2(\cos wt \cos w(t+T) + \sin wt \sin w(t+T))$$

$$= \sigma^2 \cos(wt - w(t+T)) = \sigma^2 \cos wT$$

which does not depend on t

$\Rightarrow$ So $X_3(t)$ is WSS

however it is not strictly stationary

X_3: cosinus process

Ex. 4 $\quad X_4(t) = a\sin(\omega t + \theta)$

with θ uniformly distributed over $[0, 2\pi]$
and $a, \omega \in \mathbb{R}$ const, for $t \in \mathbb{R}$

$$\mu_{X_4}(t) = E(a\sin(\omega t + \theta)) = \int_0^{2\pi} a\sin(\omega t + \varphi) f_\theta(\varphi)\, d\varphi$$

sin over one period $\therefore$ by symmetry $= 0$

$$= \int_0^{2\pi} a\sin(\omega t + \varphi)\, \tfrac{1}{2\pi}\, d\varphi = \dots = 0$$

$$R_{X_4}(t, t+\tau) = E(X_4(t)\, X_4(t+\tau)) = a^2 E(\sin(\omega t + \theta)\sin(\omega(t+\tau) + \theta))$$

$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

$$= \frac{a^2}{2} E(\cos(-\omega\tau) - \cos(2\omega t + \omega\tau + 2\theta))$$

$$= \frac{a^2}{2} E(\cos\omega\tau - 0) = \frac{a^2}{2}\cos\omega\tau$$

which does not depend on t

by symmetry of cos

So X_4 is WSS

PROPERTIES OF AUTOCOR. FCT. OF WSS PROC.

$X(t)$ WSS means that $\mu_X(t) = \mu_X$ and $R_X(t, t+\tau) = R_X(\tau)$ do not depend on t

1. Symmetric: $R_X(\tau) = R_X(-\tau)$
2. $R_X(0) = E(X(t)^2)$
3. $|R_X(\tau)| \leq R_X(0)$

proof:

1. $R_X(-\tau) = E(X(t)X(t-\tau)) = [\text{var. change } t \to t+\tau] = E(X(t+\tau)X(t+\tau-\tau))$
$= E(X(t)X(t+\tau)) = R_X(\tau)$

2. Take $\tau = 0$ in proof of 1

3. $0 \leq E((X(t+\tau) \pm X(t))^2) = E(X(t+\tau)^2) + E(X(t)^2) \pm 2E(X(t+\tau)X(t))$
$= R_X(0) + R_X(0) \pm 2R_X(\tau)$

GAUSSIAN (NORMAL) RAND. PROCESSES

$\{X(t)\}_{t\in T}$ is Gaussian random process if every lin comb $\sum_{i=1}^{n} a_i X_i(t)$ is normal distributed

$\forall\ a_1, \ldots, a_n \in \mathbb{R}\ ,\ t_1, \ldots, t_n \in T\ ,\ n \in \mathbb{N}$

Take $n=1$ $a_1=1$ $t_1=1$ to see that each $X(t)$ must be normal distr.

But this is far from being enough

Ex 3. cont

$X_3(t) = U\cos\omega t + V\sin\omega t$

now: U, V indep $N(0,\sigma^2)$ and $\omega \in \mathbb{R}$ const

Normal fct $\mu = 0$

$X_3(t)$ is a Gaussian process since

$$\sum_{i=1}^{n} a_i X_3(t) = U\sum_{i=1}^{n} a_i \cos\omega t_i + V\sum_{i=1}^{n} a_i \sin\omega t_i$$

adding two normal $\rightarrow$ new normal with $\mu = \mu_1 + \mu_2$, $\sigma^2 = \sigma_1^2 + \sigma_2^2$

$$= N\left(0,\ \sigma^2(\Sigma a_i \cos\omega t_i)^2 + \sigma^2(\Sigma a_i \sin\omega t_i)^2\right)$$

Ex. 1 cont

$\sum a_i X_1(t_i) \sim N(0, a_1^2 + \ldots + a_n^2)$ for $t_1, \ldots, t_n \in \mathbb{R}$ different

So $X_1(t)$ is Gaussian process

Ex. 2 cont

$\sum a_i X_2(t_i) = (\sum a_i)\, g \sim N(0, (\sum a_i)^2)$

So $X_2(t)$ is Gaussian process

Thm A Gaussian random process is fully probabilistically determined by its mean fct $\mu_X(t)$ & autocor/autovar fct $R_X(s,t) / K_X(s,t)$

Note: this is more or less only true for Gaussian processes.

proof check characteristic fct

$$\Psi_{X(t_1)\dots X(t_k)}(\omega_1, \dots, \omega_k) = E\left(e^{j\underbrace{(\omega_1 X(t_1) + \dots \omega_n X(t_n))}_{\text{lin comb of proc. values}}}\right)$$

$\Rightarrow N(\mu, \sigma^2)$

$$\mu = E\left[\sum_{i=1}^{n} \omega_i X(t_i)\right] = \sum_{i=1}^{n} \omega_i \mu_X(t_i)$$

$$\sigma^2 = Var\left[\sum_{i=1}^{n} \omega_i X(t_i)\right] = Cov\left[\sum_{i=1}^{n} \omega_i X(t_i) \sum \omega_k X(t_k)\right]$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \omega_i \omega_k K_X(t_i, t_k)$$

Thm A Gaussian process is strictly stationary iff it is WSS

Thm Two Gaussian process values are indep iff they are uncor.

EX. 3 cont

$$X_3(t) = U \cos \omega t + V \cos \omega t$$

now: $U, V \sim NID(0, \sigma^2)$ Normal Independent

$P(X_3(1) + 2X_3(2) \geq 3)$?

Gaussian $\Rightarrow$ $X_3(1) + 2X_3(2) \sim N(\mu, \sigma^2)$

$\mu = E\left[X_3(1) + 2X_3(2)\right] = 0$ since U & V have mean 0

$$\sigma^2 = Var\left[X_3(1) + 2X_3(2)\right] = Var\left[X_3(1)\right] + 4\,Var\left[X_3(2)\right] + 4\,Cov\left[X_3(1), X_3(2)\right]$$

$= R_{X_3}(0) + 4R_{X_3}(0) + 4R_{X_3}(1)$ = from before

$$= (5\cos\omega\cdot 0 + 4\cos\omega\cdot 1)\,\sigma^2 = \sigma^2(5 + 4\cos\omega)$$

Solution: $P(N(0, \sigma^2(5+4\cos\omega) \geq 3)$

5.4 C is about indep processes ⇔ same as our Ex 1

5.4 E discrete valued discrete time Markov chains
we will discuss in section 5.5 next week

5.4 F Gaussian processes (just covered) done with

5.4 G Ergodic processes - not in course

STATIONARY INDEP. INCREMENT PROC (5.4 D)

Def : A random process $\{X(t)\}_{t\geq 0}$ has stationary indep. increments if

- $X(0) = 0$ start at zero
- $X(t+s) - X(s)$ depends only on t (not s) for $0 \leq s \leq s+t$
  probability distr of
- $X(t+s) - X(s)$ is indep of $\{X(r)\}_{r\in[0,s]}$ for $0 \leq s \leq s+t$

(is also called Lèvy process)

Most important examples :

- poisson process :
- Winer process (Brownian motion) : non exist. derivative - white noice,
- Non-random line
  starts at zero, with some nonrand slope

For a stationary indep. incr. proc. we must have

$$\mu_X(t) = E(X(t)) = E(X(1))\, t$$

$$K_X(s,t) = Cov(X(s), X(t)) = Var(X(1)) \min(s,t)$$

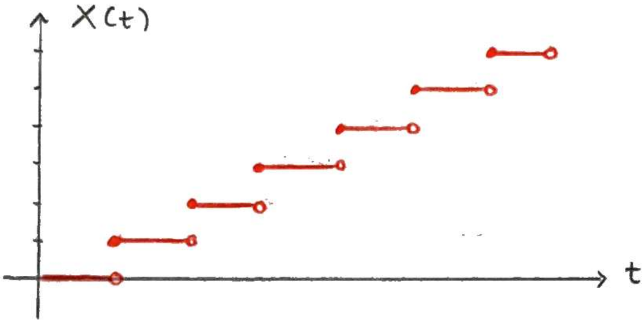
Föreläsning 3 Ch 5.6-5.7

POISSON PROCESSES

Def 1: A poisson process $(X(t))_{t\geq 0}$ with intensity/rate $\lambda > 0$ is given by

0. $X(0) = 0$
1. $X(t+s) - X(s)$ is indep. of $\{X(r)\}_{r\in[0,s]}$ for $s,t \geq 0$
2. $X(t+s) - X(s)$ is $Po(\lambda t)$ distributed for $s,t \geq 0$

Def 2: A poisson process $\{X(t)\}_{t\geq 0}$ with intensity/rate $\lambda > 0$ is given by



where $\xi_1, \xi_2, \dots$ are indep. exponential distributed with $E\{\xi_i\} = \lambda$

constructive

EX. $\mu_X(t) = E[X(1)]t = E[\underbrace{X(1)-X(0)}_{Po(\lambda)} + \underbrace{X(0)}_{0}] \cdot t = \lambda t$

or: $= E[X(t)] = E[\underbrace{X(t)-X(0)}_{Po(\lambda t)} + \underbrace{X(0)}_{0}] = \lambda t$

Autocovariance fct:

$K_X(s,t) = Cov(X(s), X(t)) = Var[X(1)] \cdot min(s,t) = Var[Po(\lambda)] \, min(s,t) = \lambda \, min($

or $\overset{s\leq t}{=} \underbrace{Cov(X(s), X(s))}_{Var[X(s)] = Var[Po(\lambda s)]} + \underbrace{Cov[X(s), X(t) - X(s)]}_{=0 \text{ since indep increments}}$

$= \lambda s$

$= \begin{cases} \lambda s & s \leq t \\ \lambda t & t \leq s \end{cases} = \lambda min(s,t)$

Ex. $P(X(1)=1 \mid X(2)=2)$

$$= \frac{P(X(1)=1,\ X(2)=2)}{P(X(2))} = \frac{P(X(2)-X(1)=1)\,P(X(1)=1)}{P(X(2))}$$

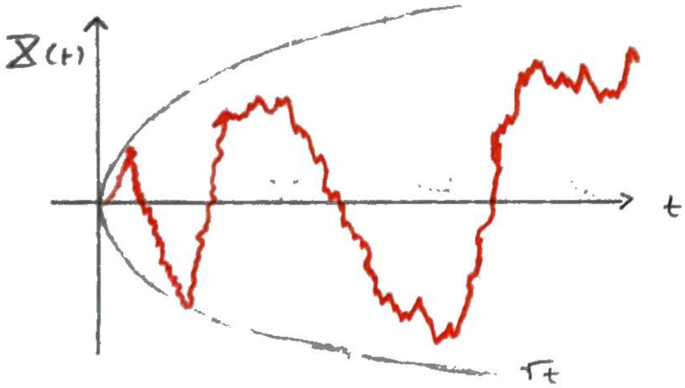
$$= \frac{P(Po(\lambda)=1)\ P(Po(\lambda)=1)}{P(Po(2\lambda)=2)} = \frac{e^{-\lambda}\frac{\lambda^1}{1!}\ e^{-\lambda}\frac{\lambda^1}{1!}}{e^{-2\lambda}\frac{(2\lambda)^2}{2!}} = \frac{1}{2}$$

WIENER PROCESSES (Brownian motion)

is a stationary independent increment process $\{X(t)\}_{t\geq 0}$ with

$$X(t+s) - X(s)$$

is $N(0,\sigma^2 t)$ distributed for $s,t \geq 0$



Not differentiable because at each s, it chooses a new dir indep of previous

Continuous fct

oscillation on infinitesimal scale

Theorem

A wiener process $\{X(t)\}_{t\geq 0}$ is a zero mean Gaussian process with

$$K_X(s,t) = R_X(s,t) = \sigma^2 \min(s,t)$$

Proof: $\mu_X(t) = E[X(t)] = E[\underbrace{X(t)-X(0)}_{N(0,\sigma^2 t)} + \underbrace{X(0)}_{0}] = E[N(0,\sigma^2 t)] = 0$

$$K_X(s,t) = Cov[X(s), X(t)] \overset{s\leq t}{=} \underbrace{Cov[X(s), X(s)]}_{Var[X(s)]} + \underbrace{Cov[X(s), X(t)-X(s)]}_{=0}$$

$$= Var[N(0,\sigma^2 t)]$$

$$= \begin{cases} \sigma^2 s & s\leq t \\ \sigma^2 t & t\leq s \end{cases}$$

To show Gaussian prossec we must show $\sum_{i=1} a_i X(t_i)$ is normal distributed for $a_1, \dots, a_n \in \mathbb{R}$, $t_1, \dots, t_n \geq 0$.

wlog: assume $0 \leq t_1 < \dots < t_n$

$$\sum_{i=1}^{n} a_i X(t_i) = a_n(X(t_n) - X(t_{n-1})) + (a_n + a_{n-1})(X(t_{n-1}) - X(t_{n-2}))$$
$$+ (a_n + a_{n-1} + a_{n-2})(X(t_{n-2}) - X(t_{n-3})) + \dots$$
$$+ (a_n + a_{n-1} + \dots + a_1) X(t_1)$$

sum of indep Gaussian fcts ?

Theorem

we can alternatively def. the wiener process $\{X(t)\}_{t\geq 0}$ to be a Gaussian process with mean fct $\mu_{X(t)} = 0$ and autocovariance fct $K_X(s,t) = \sigma^2 \min(s,t)$

Def: A wiener process with drift is a stationary indep. increment process where the requirement $X(t+s) - X(s) \sim N(0, \sigma^2 t)$ in the def of wiener process has been exchanged for $X(t+s) - X(s) \sim N(\mu t, \sigma^2 t)$

wiener process is (probabilistically) same as wiener process without drift, with μt added

MARKOV CHAINS

Def: A discrete time, discrete valued random process $\{X_n\}_{n=0}^{\infty}$ is called a Markov Chain (and has the Markov property) if

$$P_r(X_{n+1} = x_{n+1} \mid X_n = x_n, \dots, X_0(x_0)) = P_r(X_{n+1} = x_n \mid X_n = x_n)$$

* we always (more or less) assume wlog that values of Markov Chains are integers $\mathbb{Z}$

It's the same with time but we select $\mathbb{N}$

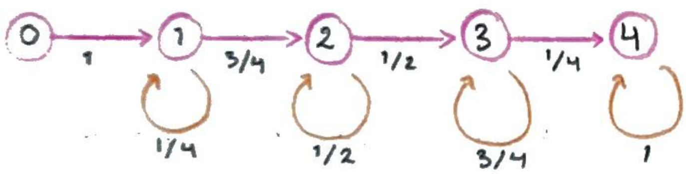
this is because they are countable, can do correspondence

Def The transition probabilities $p_{ij} = P(X_{n+1} = j \mid X_n = i)$ are elements of the transition matrix $P = (p_{ij})$

* we always (more or less) assume time homogenety :

p_{ij} does not depend on n

EX. kid collecting superhero figures at fast food restaurants



X_n = number of superheroes out of four possible collected after n restaurant visits

$X_0 = 0$, $X_1 = 1$

Is a Markov chain with
$$P = \begin{array}{cc} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \\ \begin{matrix} 0 & \;1 & \;2 & \;3 & \;4 \end{matrix} & \end{array}$$

Förelåsning 4

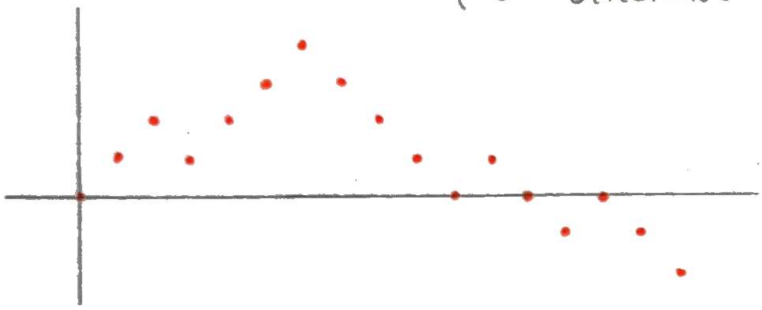
CH 5.5

Ex 2. Simple random walk

$X_0 = 0 \quad X_n = \sum_{i=1}^{n} Y_i$ where $\{Y_i\}_{i=1}^{m}$ of IID r.v

with $P\{Y_i = 1\} = p$, $P\{Y_i = -1\} = 1-p = q$ $\quad p \in (0,1)$

This gives $p_{ij} = \begin{cases} p & \text{for } j = i+1 \\ q & \text{for } j = i-1 \\ 0 & \text{otherwise} \end{cases}$

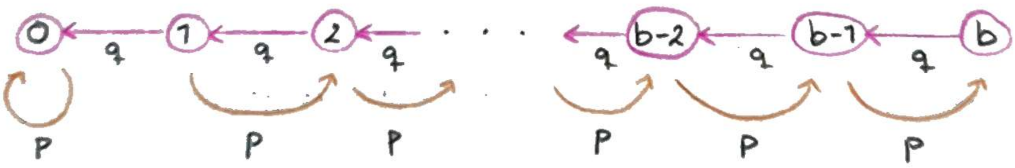


Ex. 3 Gamblers ruin

Gambler initially has $\$ X_0 = d$ while house has $\$ b-d$

He bets $\$1$ in each gamble and gets $2\$$ back with prob $p \in (0,1)$, but gets nothing $\$0$ with prob $1-p = q$

X_n is gamblers fortune after n bets



$$p_{ij} = \begin{cases} p & \text{for } j = i+1, & i \in \{1, \dots, b-1\} \\ q & \text{for } j = i-1 & i \in \{1, \dots, b-1\} \\ 1 & \text{for } j = i & i \in \{1, b\} \\ 0 & \text{otherwise} \end{cases}$$

Def: The distribution $p(n)$ at time n is the row matrix with elements $p(n)_j = P(X_n = j)$

Def: The n-step transition matrix $P^{(n)}$ has elements $p_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i)$

Theorem

$P^{(n)} = P^n$ and $p(m+n) = p(m) P^{(n)} = p(m) P^n$

proof Goal: $P^{(n+1)} = P P^{(n)}$

iteration then gives result

$$(P^{(n+1)})_{ij} = P(X_{m+n+1} = j \mid X_m = i) = \frac{P(X_{m+n+1} = j, X_m = i)}{P(X_m = i)}$$

$$= \sum_k \frac{P(X_{m+n+1} = j, X_{m+1} = k, X_m = i)}{P(X_{m+1} = k, X_m = i)} \frac{P(X_{m+1} = k, X_m = i)}{P(X_m = i)}$$

$$= \sum_k P(X_{m+n+1} = j \mid X_{m+1} = k, X_{m = i}) \, P(X_{m+1} = k \mid X_m = i)$$

$$= \sum_k (P^{(n)})_{kj} (P)_{ik} = (P P^{(n)})_{ij} = \cdots = (P^{n+1})_{ij}$$

and

$$p(m+n)_j = P(X_{m+n} = j) = \sum_k P(X_{m+n} = j, X_m = k)$$

$$= \sum_k \frac{P(X_{m+n} = j, X_m = k)}{P(X_m = k)} P(X_m = k) = \sum_k (P^{(n)})_{kj} \, p(m)_k$$

$$= (p(m) P^{(n)})_j$$

Def: A row matrix $\hat{p}$ is a stationary distribution if

1) $\hat{p} P = \hat{p}$
2) $\hat{p}_i \geq 0 \quad \forall i$
3) $\sum_i \hat{p}_i = 1$

Theorem

If $p(m) = \hat{p}$, then $p(m+n) = \hat{p} \quad \forall n \geq 1$

proof: $p(m+n) = p(m) P^n = (\hat{p} P) P^{n-1} = \hat{p} P^{n-1} = \cdots = \hat{p}$

EX 1 : $\hat{p} = (0\ 0\ 0\ 0\ 1)$

EX 2 : No stationary distr

EX 3 : $\hat{p} = (1\ 0 \cdots 0\ 0)$ or $\hat{p} = (0 \cdots 0\ 1)$

Def : The meantime to return to state i is

$\mu_i = E[\min\{n \geq 1 : X_n = i\} \mid X_0 = i]$

also called mean reccurrence time

Def : State j is accesible from state i, notation $i \to j$

if $P_{ij}^{(n)} > 0$ for some n

State i & j communicate, notation $i \leftrightarrow j$

if $i \to j$ & $j \to i$

Chain is irreducible if $i \to j$ $\forall$ states i, j

EX 1. is not irreducible

EX 2. is irreducible

EX 3. is not

Def : • period $d(i)$ of State i is $d(i) := \gcd(n \geq 1 : P_{ii}^{(n)} > 0)$

• chain is aperiodic if $d(i) = 1$ $\forall i$

EX 1 : $d(1) = d(2) = d(3) = d(4) = 1$ $d(0)$ not def

EX 2 : $d(i) = 2$ $\forall i$

EX 3 : $d(0) = d(b) = 1$ $d(1) = \ldots = d(b-1) = 2$

$E[\min\{n \geq 1 : X_n = 4\}] = E[\min\{n \geq 1 : X_n = 4 \mid X_0 = 0\}]$

$E[\text{waiting time distr. with } p=1] + E[\text{w.t.d } p = 3/4]$

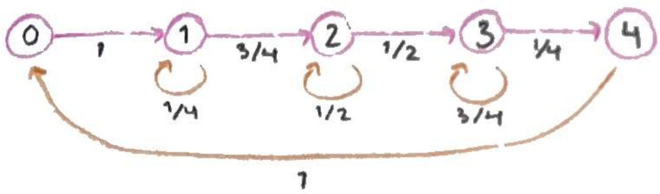
$+ E[\text{w.t.d } p = 1/2] + E[\text{w.t.d. } p = 1/4]$ Call this E

$= 1 + \frac{4}{3} + 2 + 4 = \frac{25}{3}$

Theorem

For an irreducible aperiodic Markov Chain $\hat{p}$ exists iff the mean recurrence time $\mu_i < \infty$ $\forall$ states i and in that case $\hat{p}_i = 1/\mu_i$

Ex 4: Modified superhero problem



$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 3/4 & 1/4 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$d(i) = 1 \quad \forall i$

Chain is irreducible

Does $\hat{p}$ exist?

$(\hat{p}_0 \ \hat{p}_1 \ \hat{p}_2 \ \hat{p}_3 \ \hat{p}_4) = \hat{p} P = (\hat{p}_4 \quad \hat{p}_0 + \frac{1}{4}\hat{p}_1 \quad \frac{3}{4}\hat{p}_1 + \frac{1}{2}\hat{p}_2 \quad \frac{1}{2}\hat{p}_2 + \frac{3}{4}\hat{p}_3 \quad \frac{1}{4}\hat{p}_3)$

$\Rightarrow (\hat{p}_0 \quad \frac{4}{3}\hat{p}_0 \quad 2\hat{p}_0 \quad 4\hat{p}_0 \quad \hat{p}_0)$ $\quad \Sigma = \frac{28}{3}$

$\Rightarrow \hat{p}_0 = \frac{3}{28}$

So $\hat{p} = (\frac{3}{28} \ \ \frac{4}{28} \ \ \frac{6}{28} \ \ \frac{12}{28} \ \ \frac{3}{28})$

$E = \mu_0 - 1 = \frac{28}{3} - 1 = \frac{25}{3}$ Good

Def • State i is transient if

$$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) < 1$$

• State i is reccurent/persistent if

$$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1$$

Ex 1. 0, 1, 2, 3 transient , 4 recurrent

Ex 2. Remains to be determined

Ex 3. 0, b reccurent , 1, ..., b-1 transient

Ex 4. 0, 1, 2, 3, 4 recurrent

Theorem

i is recurrent iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$

Theorem

For an irreducible either all states are recurrent or all are transient.

All states have same period

Further, $\mu_i < \infty$ for some state i iff $\mu_i < \infty$ $\forall$ states i

Ex 2. Is simple random walk recurrent ?

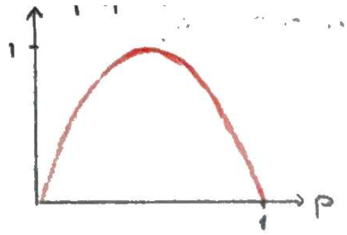
I.e is $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$ or not ?

$$P_{ii}^{(n)} = \begin{cases} p^k(1-p)^k \binom{2k}{k} & n \text{ even } = 2k \\ 0 & n \text{ odd} \end{cases}$$

$$\sum \binom{2k}{k} p^k(1-p)^k = \sum \frac{(2k)!}{k!\,k!} p^k(1-p)^k = \infty$$

[Sterlings formula : $k! \approx \sqrt{2\pi k}\, k^k e^{-k}$ $\quad k \to \infty$]

$$= \sum_{k=1}^{\infty} \frac{\sqrt{4\pi k}\,(2k)^{2k}\,\cancel{e^{-2k}}}{2\pi k\, k^{2k}\,\cancel{e^{-2k}}} p^k(1-p)^k = \sum \frac{(4p(1-p))^k}{\sqrt{\pi k}}$$



$\Sigma = \infty$ for $p = 1/2$

$\Sigma < \infty$ for $p \neq 1/2$

$\Rightarrow$ Recurrence for $p = 1/2$

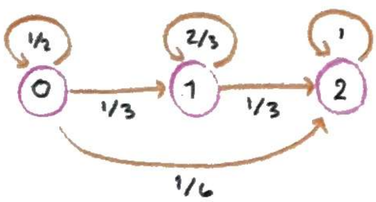
Transient for $p \neq 1/2$

COMPUTER PROBLEM

for own work Exercise Session 1

Time homogeneous Markov Chain $\{X_n\}_{n=0}^{\infty}$ with state space E, initial distribution $p(0)$ and transition matrix P given by

$E = \{0, 1, 2\}$ $\quad p(0) = [1 \ 0 \ 0]$ & $\quad P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$



Find $E[T]$ for $T = \min\{n \geq 1 : X_n = 2\}$ by statistical simulation

$$E_0 = 1 + \tfrac{1}{2} E_0 + \tfrac{1}{3} E_1 \qquad \Rightarrow \qquad \begin{cases} E_0 = 4 \\ E_1 = 3 \end{cases}$$
$$E_1 = 1 + \tfrac{2}{3} E_1$$

Analytical sol

But this is not allowed, you have to program

In[1]. Matematica

```
For [i=1 ; Rep = 100 000 ; Time = 0 , i ≤ Rep , i++
    X = 0
    While [X = 0 , Time = Time+1 ; Chance = Random [] ;
        If [Chance < 1/6 , X = 2 , If [Chance < 1/2 , X = 1 ] ] ] ;
```

Random uniform dist between 0 & 1

```
While [x = 1 , Time = Time + 1 ; Chance = Random [] ;
    If [Chans < 1/8 , x = 2]]] ;
N [Time/Rep]
Out [1] = 4.00004
```

Förelāsning 5

CH 5.8, 6.1-6.3B 9

Suppose that a random process $\mathbb{X}(t)$ is WSS with ACF

$$R_X(t, t+\tau) = e^{-|\tau|/2} = R_{\mathbb{X}}(t)$$

$$E(\mathbb{X}(t), \mathbb{X}(t+\tau))$$

a) Find second moment of $\mathbb{X}(5)$

b) ——— " ——— of $\mathbb{X}(5) - \mathbb{X}(2)$

Solution :

$$E[\mathbb{X}(t)^2] = R_{\mathbb{X}}(0) = 1$$

$$E[(X(5)-X(3))^2] = E[X(5)^2] + E[X(3)^2] - 2E[X(3)X(5)] = 2(1-e^{-1})$$

EX 5.86

Consider random process $X(t)$ given by

$$X(t) = U\cos t + (V+1)\sin t \qquad t \in \mathbb{R}$$

where U & V are indep r.v. with $E(U) = E(V) = 0$ &
$E(U^2) = E(V^2) = 1$

a) Find autocovariance function of $X(t)$

b) Is $X(t)$ WSS ?

Solution :

$$K_X(s,t) = \mathrm{Cov}(X(s), X(t)) = \mathrm{Cov}(U\cos s + (V+1)\sin s,\ U\cos t + (V+1)\sin$$

$$= \mathrm{Var}(U)\cos s\cos t + \mathrm{Cov}(U, V+1)\cos s\sin t$$

$$+ \mathrm{Cov}(V+1, U)\sin s\cos t + \mathrm{Var}(V+1)\sin s\sin t$$

$$= 1\cdot\cos s\cos t + 0\,\cancel{\cos s\sin t} + 0\,\cancel{\sin s\cos t} + 1\sin s\sin t$$

Covariance of indep is 0

WSS also requires mean is const

$\mu_x(t) = E[X(t)] = E(U)\cos t + E[V+1]\sin t = \sin t$

so not WSS

MARTINGALES

In basic probability $E[Y \mid X_0 = x_0, \dots, X_n = x_n]$

$$= \begin{cases} \int_{-\infty}^{\infty} \frac{f_{Y,X_0,\dots X_n}(y, x_0, \dots x_n)}{f_{X_0,\dots X_n}(x_0, \dots, x_n)}\, y\, dy \\ \sum_{k=-\infty}^{\infty} \frac{f_{Y,X_0,\dots X_n}(k, x_0, \dots, x_n)}{f_{X_0,\dots,X_n}(x_0, \dots, x_n)}\, k \end{cases} = g(x_0, \dots, x_n)$$

In advanced probability $E[Y \mid X_0, \dots, X_n] = g(X_0, \dots, X_n)$

We use F_n to denote the information $X_0, \dots, X_n$
which is also denoted $\sigma(X_0, \dots, X_n)$

$E[Y \mid X_0, \dots X_n] = E[Y \mid F_n] = E[Y \mid \sigma(X_0, \dots, X_n)] = g(X_0, \dots, X_n)$

with $g: \mathbb{R}^{n+1} \to \mathbb{R}$ from above

Def: a r.v Y is called F_n-measurable if Y is a function of $X_0, \dots, X_n$

Theorem

1) $E[aY_1 + bY_2 \mid F_n] = a\,E[Y_1 \mid F_n] + b\,E[Y_2 \mid F_n]$

2) $E[Y \mid F_n] \geq 0$ for $Y \geq 0$

3) $E[Y \mid F_n] = Y$ for Y F_n measurable

4) $E[ZY \mid F_n] = Z\,E[Y \mid F_n]$ for Z F_n measurable

5) $E[Y \mid F_n] = E[Y]$ for Y indep of F_n

6) $E[E[Y \mid F_n] \mid F_m] = E[Y \mid F_m]$ for $0 \le m \le n$
towering

7) $E[E[Y \mid F_n]] = E[Y]$

8) $E[g(Y) \mid F_n] \ge g(E[Y \mid F_n])$ for $g: \mathbb{R} \to \mathbb{R}$ convex fct
Jensens inequality

proof of 7)

$$E[g(X_0,\dots,X_n)] = \iint_{\substack{(x_0,\dots,x_n)\\ \in \mathbb{R}^{n+1}}} \int_{-\infty}^{\infty} y \, \frac{f_{Y,X_0,\dots,X_n}(y,x_0,\dots,x_n)}{\cancel{f_{X_0,\dots,X_n}(x_0,\dots,x_n)}} \, dy \, \cancel{f_{X_0,\dots,X_n}}(x_0,\dots,x_n) \, dx_0 \dots dx_n$$
$$= E[Y]$$

- Until further notice $\{X_n\}_{n=0}^{\infty}$ is a discrete time process
- we call $\{F_n\}_{n=0}^{\infty}$ a filtration

Def $\{X_n\}_{n=0}^{\infty}$ is a martingale / sub-m. / super-m. if

$E[|X_n|] < \infty \quad \forall n$

$$E[X_{n+1} \mid F_n] \begin{array}{ll} \ge X_n & \text{sub} \\ = X_n & \\ \le X_n & \text{super} \end{array}$$

Theorem

$\{X_n\}_{n=0}^{\infty} \Rightarrow E[X_n] = E[X_0] \quad \forall n$

$E[X_{m+n} \mid F_n] = X_n \quad \forall \; m \ge 1$

$\{g(X_n)\}_{n=0}^{\infty}$ is a submartingale for $g: \mathbb{R} \to \mathbb{R}$ convex proof $E[X_n] = E[E[X_{n+1} \mid F_n]] \overset{\cdot}{=} E[X_{n+1}]$

iterate to X_0

$E[X_{m+n} \mid F_n] \overset{6}{=} E[E[X_{m+n} \mid F_k] \mid F_n]$ $k \geq n$ choose $k=m$.

$= E[E[X_{m+n} \mid F_{m+n-1}] \mid F_n] = E[X_{m+n-1} \mid F_n]$

$= \dots X_n$

$E[g(X_{n+1}) \mid F_n] \overset{8}{\geq} g(E[X_{n+1} \mid F_n]) = g(X_n)$

Note: Hsu focuses a lot on verifying $E[|X_n|] < \infty$

BUT it is enough to check second property in def
(at least for martingale issues)

conditional expectation in advanced sense only exists
when means are finite

A $\{0, 1, \dots, +\infty\}$ - valued r.v T is called a stopping time if
$\{T = n\}$ is F_n - measurable
(or equiv $\{T \leq n\}$ is F_n - measurable)

Theorem

optional stopping theorem

For $\{X_n\}_{n=0}^{\infty}$ a martingale & T a stopping time we have

$$E[X_T] = E[X_0]$$

under the following conditions:

1) $E(T) < \infty$

2) $E(|X_t|) < \infty$

3) $\lim_{n \to \infty} E[|X_n| 1_{\{T > n\}}] = 0$

Ex. computational task 3

Let $X_0 = 100$ and $X_n = \sum_{i=1}^{n} Y_i \quad n \geq 1$

where $\{Y_i\}_{i=1}^{\infty}$ are IID r.v with $P(Y_1 = 4) = 1/5$, $P(Y_1 = -1) = 4/5$

Then $\{X_n\}_{n=1}^{\infty}$ is a martingale since

$$E[X_{n+1} \mid F_n] = E[X_{n+1} \mid \sigma(X_0, \ldots, X_n)] = E[Y_{n+1} + X_n \mid \sigma(X_0 \ldots X_n)]$$
$$\overset{5,3}{=} E[Y_{n+1}] + X_n = X_n$$

Now $T = \min\{n \geq 1 : X_n = 0 \text{ or } X_n \geq 200\}$

If OST applies then $100 = E[X_0] = E[X_T]$

$$= E[X_T 1_{\{X_T = 0\}} + X_T 1_{\{X_T \geq 200\}}]$$
$$= 0 \cdot P(X_T = 0) + \{200, 201, 202, 203\} \cdot f(X_T = 200)$$

$$\Rightarrow \quad P(X_T \geq 200) = \left[\frac{100}{203}, \frac{100}{200}\right]$$

what about conditions ①-③ of OST

1) In a way heuristically obvious - see course web page

2) $E[|X_T|] \leq E[203] < \infty$

3) $E[|X_n| 1_{\{T > n\}}] \leq 199 \, P(T > n) \longrightarrow 0$

Föreläsning 6

CONTINUOUS TIME MARTINGALES

Def. $\{X(t)\}_{t\geq 0}$ is a sub/super/martingale wrt $F_t = \sigma(\{X(s)\}_{s\in[0,t]})$ if $E[|X(t)|] < \infty$ and $E[X(t) \mid F_s] \leq / = / \geq X(s)$ for $0 \leq s \leq t$

EX. Wiener process (Brownian motion)

$\{X(t)\}_{t\geq 0}$ stationary indep increments process with $X(t+s) - X(s) \sim N(0, \sigma^2 t)$ gives

$$E[|X(t)|] = E[|X(t) - X(0)|] = E[|N(0, \sigma^2 t)|] < \infty \quad \text{and}$$

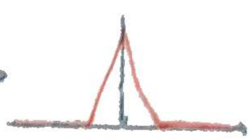
$$E[X(t) \mid F_s] = E[\underbrace{X(t) - X(s)}_{\text{indep of } F_s} \mid F_s] + E[\underbrace{X(s)}_{F_s \text{ measurable}} \mid F_s] = 0 + X(s) \quad 0 \leq s \leq t$$

COMPUTER PROBLEM

Let $\{W(t)\}_{t\geq 0}$ be a wiener process with $\sigma^2 = \text{Var}\{W(1)\} = 1$

For a real $\varepsilon > 0$ consider the differential ratio process $\{\Delta_\varepsilon(t)\}_{t\geq 0}$ given by $\Delta_\varepsilon(t) = \frac{1}{\varepsilon}(W(t+\varepsilon) - W(t)) \quad t \geq 0$

Show that the ACF $R_{\Delta_\varepsilon}(t) = R_{\Delta_\varepsilon}(s, s+t) = E[\Delta_\varepsilon(s)\, \Delta_\varepsilon(s+t)]$ of $\Delta_\varepsilon(t)$ is a triangle like function that depends on the $R_{\Delta_\varepsilon}(t)$ difference t between s & $s+t$ only and that

$R_{\Delta_\varepsilon}(t) \longrightarrow$  ie $R_{\Delta_\varepsilon}(t) \to \delta(t)$ as $\varepsilon \downarrow 0$

Solution:

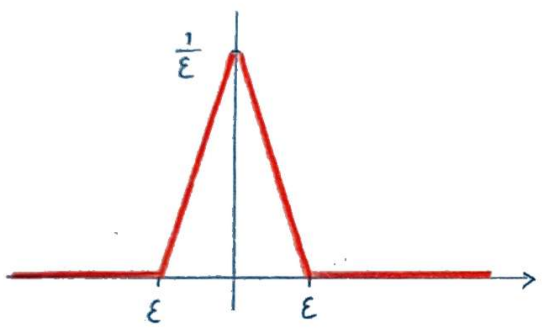
$$\mu_{\Delta_\varepsilon}(t) = E[\Delta_\varepsilon(t)] = E\left[\frac{W(t+\varepsilon)-W(t)}{\varepsilon}\right] = 0$$

$$R_{\Delta_\varepsilon}(s, s+t) = E[\Delta_\varepsilon(s)\,\Delta_\varepsilon(s+t)] = E\Big[\frac{W(s+\varepsilon)-W(s)}{\varepsilon} \quad \frac{W(s+t+\varepsilon)-W(s+t}{\varepsilon}$$

$$= \frac{1}{\varepsilon^2}\big(\min(s+\varepsilon, s+t+\varepsilon) - \min(s+\varepsilon, s+t) - \min(s, s+t+\varepsilon) + \min(s, s+t$$

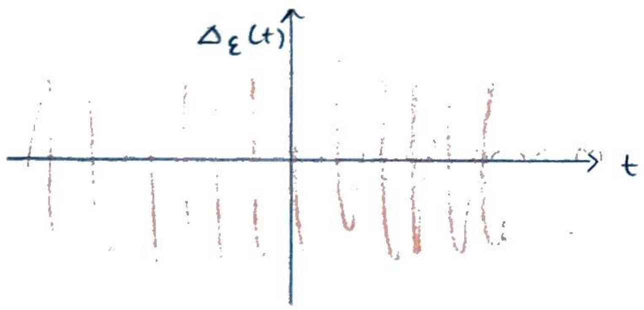
$$= \frac{1}{\varepsilon^2}\big(\min(\varepsilon, t+\varepsilon) - \min(\varepsilon, t) - \min(0, t+\varepsilon) + \min(0, t)\big)$$

$$= \frac{1}{\varepsilon^2}\begin{cases} \varepsilon - \varepsilon + 0 + 0 = 0 & t > \varepsilon \\ \varepsilon - t + 0 + 0 = \varepsilon - t & 0 \le t \le \varepsilon \\ t+\varepsilon - t - 0 + t = t+\varepsilon & -\varepsilon \le t \le 0 \\ t+\varepsilon - t - (t+\varepsilon) + t = 0 & t < -\varepsilon \end{cases}$$



process values further than ε apart are completely indep. since Gaussian & uncorrelated

$$Var(\Delta_\varepsilon(t)) = R_{\Delta_\varepsilon}(0) - R_{\Delta_\varepsilon}(t,t) = 1/\varepsilon$$



QUEUING THEORY HSU CH 9

Preparation for ch 9 about exponential distribution

Def: a continuous r.v $T > 0$ is exp. distributed with parameter $\lambda > 0$ if it has PDF

$$f_T(t) = \begin{cases} 0 & t < 0 \\ \lambda e^{-\lambda t}; & t \ge 0 \end{cases}$$

Note: In some sources : $f_T(t) = \begin{cases} 0 & t<0 \\ 1/\lambda \, e^{-t/\lambda} & t \geq 0 \end{cases}$

THM 1

If $T_1, \ldots, T_n$ are indep $\exp(\lambda_1), \ldots, \exp(\lambda_n)$ respectively

then $\min(T_1, \ldots, T_n)$ is $\exp(\lambda_1 + \ldots + \lambda_n)$

THM 2

If $T_1, \ldots, T_n$ are indep $\exp(\lambda_1), \ldots, \exp(\lambda_n)$ resp.

then $P(\min(T_1, \ldots, T_n) = T_i) = \dfrac{\lambda_i}{\lambda_1 + \ldots + \lambda_n}$

THM 3

If $T \sim \exp(\lambda) \Rightarrow P(T > t+s \mid T > s) = P(T > t)$ $\quad s,t \geq 0$

Lack of memory

THM 3 only for $\exp(\lambda)$

Proof THM 1 Remember $T \sim \exp(\lambda)$ means.

$$P(T>t) = \int_t^\infty \lambda e^{-\lambda s} \, ds = \left[-e^{-\lambda s}\right]_t^\infty = e^{-\lambda t}$$

$$P(\min(T_1, \ldots, T_n) > t) = P(T_1 > t, \ldots, T_n > t)$$

$$\overset{\text{indep}}{=} P(T_1 > t) \cdots P(T_n > t) = e^{-\lambda_1 t} \cdots e^{-\lambda_n t} = e^{-(\lambda_1 + \ldots + \lambda_n)t}$$

$$\Rightarrow \min(T_1, \ldots, T_n) \sim \exp(\lambda_1 + \ldots + \lambda_n)$$

Proof THM 2 $P(\min(T_1, \ldots, T_n) = T_i) = P(\min(\underbrace{\min(T_1, \ldots, T_{i-1}, T_{i+1}, \ldots)}_{\exp(\lambda_1 + \ldots + \lambda_{i-1} + \lambda_{i+1} + \ldots)}, \underbrace{T_i}_{\exp(\lambda_i)}) = T_i$

prove to equal $\dfrac{\lambda_i}{(\lambda_1 + \ldots + \lambda_{i-1} + \lambda_{i+1} + \ldots) + \lambda_i}$

Means that it is sufficient to prove thm for $n=2$

$$P(\min(T_1, T_2) = T_1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$ prove this

$$P(\min(T_1, T_2) = T_1) = P(T_1 \leq T_2)$$

$$= \iint_{0 \leq x \leq y < \infty} F_{T_1 T_2}(x,y)\, dx\, dy = \int_{x=0}^{\infty} \Big[\underbrace{\int_{y=x}^{\infty} \lambda_2 e^{-\lambda_2 y}\, dy}_{e^{-\lambda_2 x}} \Big] \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \int_{x=0}^{\infty} \lambda_1 e^{-(\lambda_1 + \lambda_2)x}\, dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

proof THM 3 $\quad P(T > t+s \mid T > s) = \dfrac{P(T > t+s,\ T > s)}{P(T > s)} \qquad s, t \geq 0$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = P(T > t)$$

converse : THM 3 true means $\dfrac{P(T > t+s)}{P(T > s)} = P(T > t)$

Functional equation.

only pos. sol. $P(T > t) = e^{ct}$ wheras prob $c \leq 0$

PREPARATION FOR BIRTH / DEATH PROCESSES CH 9

* A birth-and death process is an $\mathbb{N}$-valued $\{X(t)\}_{t \geq 0}$ s.t

- X_0 has a certain random or nonrandom value in $\mathbb{N}$
- when $X(t)$ gets a certain value $n \in \mathbb{N}$ it stays at that value a $\min(\exp(\mu_n, \lambda_n)) = \exp(\mu_n + \lambda_n)$ time after which the value changes to
  - $n-1$ if $\exp(\mu_n) = \min(\exp(\mu_n), \exp(\lambda_n))$ with prob $\mu_n / (\mu_n + \lambda_n)$
  - $n+1$ if $\exp(\lambda_n) =$ — " — with prob $\lambda_n / (\mu_n + \lambda_n)$

  $\mu_0 = 0, \quad \lambda_1, \mu_1, \lambda_2, \mu_2, \dots \geq 0$

  $\exp(\mu_n), \exp(\lambda_n)$ indep
- EX : Poisson process $\mu_0 = \mu_1 = \dots = 0, \quad \lambda_1 = \lambda_2 = \dots = \lambda$

Graphical Display of Birth & death processes :

parameters

Queuing system will be a special case of birth-and death-process (described next week)

We will do analytical calculations for birth-and death process (queuing systems) in steady state only

So that $P(X(t)=n)$ doesn't depend on n

Steady state: flow out of state = flow into state

$$\begin{cases} P(X(t)=n)(\lambda_n+\mu_n) = P(X(t)=n-1)\lambda_{n-1} + P(X(t)=n+1)\mu_{n+1} & n \geq 1 \\ P(X(t)=0)\lambda_0 = P(X(t)=1)\mu_1 \end{cases}$$

Difference equation of order 2. inhomogeneous

Solution: $P_n = P_0 \dfrac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}$ where $P_0 = \left(1 + \sum_{n=1}^{\infty} \dfrac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}\right)^{-1}$

CONTINUITY

Def: $X_n \to X$ in mean square denoted $\underset{n\to\infty}{\text{l.i.m}}\, X_n = X$

if $E[X^2] < \infty$ and $E[(X_n - X)^2] \to 0$ as $n \to \infty$

Thm

$\underset{n\to\infty}{\text{l.i.m}}\, X_n = X$ and $\underset{n\to\infty}{\text{l.i.m}}\, Y_n = Y$

$$\Rightarrow \begin{cases} \underset{n\to\infty}{\text{l.i.m}}\, E[X_n] = E[X] \\ \underset{n\to\infty}{\text{l.i.m}}\, E[X_n^2] = E[X^2] \\ \underset{n\to\infty}{\text{l.i.m}}\, E[X_nY_n] = E[XY] \end{cases}$$

Def: A continuous time process $X(t)$ is cont at time $t=t_0$ if

$\underset{t\to t_0}{\text{l.i.m}}\, X(t) = X(t_0)$

Thm

$X(t)$ is cont. $t=t_0$ if $R_x(s,t)$ is cont at $(s,t) = (t_0, t_0)$

and then $\mu_x(t)$ is cont at $t=t_0$

proof $E[(X(t)-X(0))^2] = R_x(t,t) - 2R_x(t,t_0) + R_x(t_0,t_0) \longrightarrow 0$

for $R_x(s,t)$ cont at $(s,t) = (t_0, t_0)$

If so : $|\mu_x(t) - \mu_x(t_0)| = |E[X(t)] - E[X(t_0)]| \leq E[|X(t) - X(t_0)|]$

triangle ineq for exp. value

$\underset{\text{C.S.}}{\leq} \sqrt{E[(X(t)-X(t_0))^2]} \longrightarrow 0$

EX. Stationary indep increment processes are cont. as

$R_x(s,t) = k_x(s,t) + \mu_x(s)\mu_x(t) = Var[X(1)]\min(s,t) + E[X(1)]s\,E[X(1)]t$

is continuous

DIFERENTIABILITY

Def : a cont time process $X(t)$ is diff.ble at $t=t_0$ with derivative $X'(t_0)$ if

$$\underset{t\to t_0}{\text{l.i.m}} \frac{X(t)-X(t_0)}{t-t_0} = X'(t_0)$$

Thm

$X(t)$ is diff.ble at $t=t_0$ if $\dfrac{\partial^2 R_x(s,t)}{\partial s\,\partial t}$ exists at all (s,t)

and then $R_x(s,t) = \dfrac{\partial^2 R_x(s,t)}{\partial s\,\partial t}$

EX. Stationary indep increment processes

$$\frac{\partial R_x(s,t)}{\partial s} = \begin{cases} Var(X(1)) & s\leq t \\ 0 & s>t \end{cases} + E[X(1)]^2 t$$

not continuous, so not differentiable

$\Rightarrow$ Process not diff.ble

INTEGRALS

$$\int_a^b X(t)\,dt = \text{l.i.m} \left\{ \sum_{i=1}^{n} X(s_i)(t_i - t_{i-1}) : a = t_0 < t_1 \ldots < t_n = b, \; s_i \in [t_{i-1}, t_i], \; \max_{1 \le i \le n} t_i - t_{i-1} \searrow 0 \right\}$$

Thm

$$E\left[\int_a^b X(s)\,ds \cdot \int_c^d Y(t)\,dt\right] = \int_a^b \int_c^d \underbrace{R_{XY}(s,t)}_{= E[X(s)Y(t)]} \,ds\,dt$$

cross correlation

proof integral $\rightarrow$ sum $\rightarrow$ lift out sum $\rightarrow$ integral

CROSS CORELATION FCT

Def: Two WSS processes $X(t)$ and $Y(t)$ are jointly WSS if

$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$

depends on τ only and not t.

If so, write $R_{XY}(\tau)$

Thm

For $X(t)$ and $Y(t)$ jointly WSS we have

1. $R_{XY}(\tau) = R_{XY}(-\tau)$
2. $|R_{XY}(\tau)| \le \sqrt{R_X(0)\,R_Y(0)}$
3. $|R_{XY}(\tau)| \le \frac{1}{2}[R_X(0) + R_Y(0)]$

proof 1 $R_{XY}(\tau) = E[Y(t)X(t+\tau)] = E[X(t+\tau)Y(t)] = R_{XY}(-\tau)$

3 $-2\sqrt{R_X(0)\,R_Y(0)} + R_X(0)\,R_Y(0) = \left(\sqrt{R_X(0)} - \sqrt{R_Y(0)}\right)^2 \ge 0$

proof 2

$$0 \le E\left[\left(\frac{X(t)}{\sqrt{R_X(0)}} \pm \frac{Y(t+\tau)}{\sqrt{R_Y(0)}}\right)^2\right] = 1 + 1 \pm 2\,\frac{R_{XY}(\tau)}{\sqrt{R_X(0)\,R_Y(0)}}$$

Förelāsning 7

Today : Chapter 9 in HSU + computational Task Exercise sheet 3

QUEUING SYSTEM

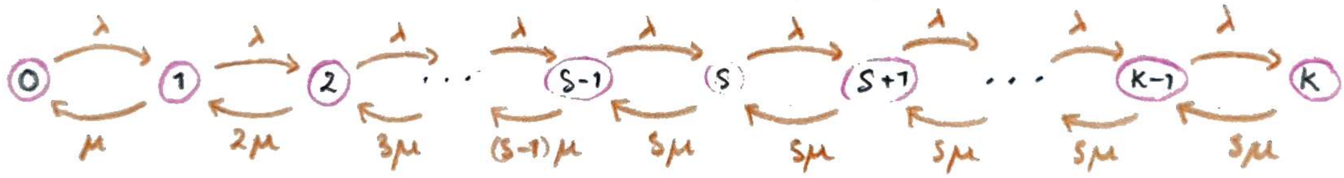
Denoted $M/M/S/K$ or $M(\lambda)/M(\mu)/S/K$

$\lambda, \mu > 0$ $\quad S \in \{1, 2, \ldots\}$ $\quad K \in \{S, S+1, \ldots, +\infty\}$

* N.o tot customers $\{X(t)\}_{t\geq 0}$ in queing system is a birth & death process with

$$\begin{cases} \lambda_n = \lambda & \text{for } n = 0, 1, \ldots, K-1 \\ \lambda_n = 0 & \text{for } n \geq K \end{cases}$$

$$\begin{cases} \mu_n = n\mu & \text{for } n \in \{1, \ldots, S\} \\ \mu_n = S\mu & \text{for } n \in \{S+1, \ldots, K\} \end{cases}$$



* New customers arrive with indep $\exp(\lambda)$ interarrival time

* System has S servers that require indep $\exp(\mu)$ times

* System has $K-S$ queing slots where customers wait for service if all servers are busy

* If $K < \infty$ customers that (try to) arrive to system when full $X(t) = K$ "bounce away" and disappears

* when $X(t)$ gets value n next value is $n-1$ or $n+1$ depending on which happens first (ie is smallest) of an $\exp(\mu_n)$ time until first server finishes serving and $\exp(\lambda_n)$ time until next arrival of new customer

* with that competition finished a new analogue competition starts for the value after that where exp-times from previous competition are forgotten

* Each competition for next value after $X(t)=n$ last an $\exp(\lambda_n+\mu_n)$ - time after which

$$X(t) = \begin{cases} n-1 & \text{with prob. } \mu_n/(\mu_n+\lambda_n) \\ n+1 & \text{with prob. } \lambda_n/(\mu_n+\lambda_n) \end{cases}$$

* we do analytical calculation for queuing systems when they have steady state / equilibrium distribution

* $$P(X(t)=n) = P_n = P_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$$

* Average time between arriving customers is $E[\exp(\lambda)] = \frac{1}{\lambda}$
So on average λ customers try to arrive to queuing system per unit time

* For $K<\infty$ not all customers that try to arrive to queing system really are let in (system can be full) and therefore we have that on average $\lambda_e = \lambda_{\text{efficient}} = \begin{cases} \lambda(1-p_K) & K<\infty \\ \lambda & K=\infty \end{cases}$
customer joins system per time unit.

* Beware: Hsu has made some errors with usage of λ_e so in Ch 9. Consult errata list

Def: Trafic intensity $\rho = \frac{\lambda}{s\mu}$

Thm

$\rho < 1$ nec. for equilibrium to be possible if $K = \infty$

we will be interested in six quantities (except for state probabilities p_n)

- L = mean number of customers in whole queuing system
- L_q = —— " —— queuing for service of queuing slots
- L_s = —— " —— that are being served
- W = mean time customers spend in whole queuing system
- W_q = —— " —— queuing before service
- W_s = —— " —— being served

Thm

$$L = \sum_{n=0}^{k} n\, P(X(t) = n) = \sum_{n=0}^{k} n\, p_n$$

$$L = L_q + L_s$$

$$W = W_q + W_s$$

$$L_q = \lambda_e W_q \qquad L_s = \lambda_e W_s$$

$$W_s = 1/\mu$$

proof By inspection

Consequence

$W_s \Rightarrow L_s = \lambda_e W_s$

$L_s \Rightarrow L_q = L - L_s$

$L_q \Rightarrow W_q = L_q \lambda_e$

$W_s, W_q \Rightarrow W$

EX. $M(\lambda)/M(\mu)/1 \qquad (s=1,\ k=\infty,\ 0<\lambda<\mu)$

$$L = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} n p_0 \left(\frac{\lambda}{\mu}\right)^n = \sum n \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

p_0 is selected so $\sum_{n=0}^{\infty} p_n = 1$

$$= \sum_{n=1}^{\infty} (n-1)\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n-1} = \frac{1}{1-\frac{\lambda}{n}} - 1 = \frac{\lambda}{\mu-\lambda}$$

$$W_s = 1/\mu$$

$$L_s = \lambda_e W_s = \lambda \frac{1}{\mu} = \frac{\lambda}{\mu}$$

$$L_q = L - L_s = \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu} = \frac{\lambda^2}{(\mu-\lambda)\mu}$$

$$W_q = \frac{L_q}{\lambda_e} = \frac{\lambda^2}{(\mu-\lambda)\mu}\cdot\frac{1}{\lambda} = \frac{\lambda}{(\mu-\lambda)\mu}$$

$$W = W_s + W_q = \frac{\lambda}{(\mu-\lambda)\mu} + \frac{1}{\mu} = \frac{1}{\mu-\lambda}$$

COMPUTATIONAL TASK EX. SES. 3

$M_0 = 100 \qquad M_n = 100 + \sum_{i=1}^{n} X_i$

$\{X_i\}_{i=1}^{\infty}$ are iid with $P(X_i=-1) = 4/5 \qquad P(X_i=4) = 1/4$

So $E[X_i] = 0$

$\{M_n\}_{n=0}^{\infty}$ Martingale wrt $F_n = \sigma(X_0, \dots, X_n) = \sigma(M_0, \dots, M_n)$

since $E[M_{n+1} \mid F_n] = E[X_{n+1} + M_n \mid F_n] = E[\cancel{X_{n+1}}] + M_n = M_n$

we run M_n to time $T = \min\{n \geq 0 : M_n = 0 \text{ or } M_n \geq 200\}$

Solve by computer simulation.

```
In[1] = Reps = 1 000 000 ;
M0 = 100 ;
For [i = 1 ; sucess = 0 , i ≤ Reps , i++,
    M = M0
```

```
    While [0 < M && M < 200,
        If [Random[] ≤ 1/5 , M = M+4 , M = M-1 ]];
    If [M ≥ 200 , Success = Success + 1 ]];
N [Success / Reps ]

Out[1] = 0.493
```

Föreläsning 8 — CRASH COURSE 2

- Fourier transform (= mathematical version of CHF)
- δ - function in discrete & cont. time
- Convolutions

FOURIER TRANSFORM CONT.

1-dim F-trans. of $f : \mathbb{R} \to \mathbb{C}$

$$\hat{f}(\omega) = (Ff)(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x} f(x)\, dx$$

$j^2 = -1$

1-dim F-inverse-trans.

$$f(x) = (F\hat{f})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega x} \hat{f}(\omega)\, d\omega$$

n-dim F-trans of $f : \mathbb{R}^n \to \mathbb{C}$

$$\hat{f}(\omega_1, \dots, \omega_n) = (Ff)(\bar{\omega}) = \int_{-\infty}^{\infty} e^{-j\bar{\omega}\cdot\bar{x}} f(\bar{x})\, d\bar{x}$$

n-dim F-inverse-trans

$$f(x_1, \dots, x_n) = (F\hat{f})(\bar{x}) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{j\bar{\omega}\cdot\bar{x}} \hat{f}(\bar{\omega})\, d\bar{\omega}$$

Ex 1. $f(x) = f_{N(\mu,\sigma^2)}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$\Rightarrow \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu+j\omega\sigma^2)^2}{2\sigma^2}} dx \cdot e^{-j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$$

$$= e^{-j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$$

Still normal distr but other Exp val & var.

Still integrates to 1

$\varphi_{N(\mu,\sigma^2)}(\omega) = e^{j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$

characteristic fct.

Ex 2. $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} = f_{exp(\lambda)}(x)$

$$\hat{f}(w) = \int_{-\infty}^{\infty} e^{-jwx} \lambda e^{-\lambda x} \, dx = \lambda \left[\frac{-1}{jw+\lambda} e^{-(jw+\lambda)x} \right]_0^{\infty}$$

$$= \frac{\lambda}{jw+\lambda}$$

$$\varphi_{exp(\lambda)}(w) = \frac{\lambda}{-jw+\lambda}$$

Properties

1. $F(f(\cdot - x_0))(w) = e^{-jwx_0} (Ff)(w)$ translation

2. $F(e^{jw_0} f)(w) = (Ff)(w-w_0)$ mult. with complex

3. $(Ff(-x))(w) = (Ff)(-w)$ mirroring

4. $(Ff')(w) = jw(Ff)(w)$ Fourier of derivative

proof 1 $\int_{-\infty}^{\infty} e^{-jwx} f(x-x_0)\,dx = [y = x - x_0] = \int_{-\infty}^{\infty} e^{-jw(y+x_0)} f(y)\,dy$

$= e^{-jwx_0} (Ff)(w)$

2 $\int_{-\infty}^{\infty} e^{-jwx} e^{jw_0 x} f(x)\,dx = \int_{-\infty}^{\infty} e^{-j(w-w_0)x} f(x)\,dx = (Ff)(w-w_0)$

3 $\int_{-\infty}^{\infty} e^{-jwx} f(-x)\,dx = [y = -x] = \int_{-\infty}^{\infty} e^{jwy} f(y)\,dy = (Ff)(-w)$

4) $\int_{-\infty}^{\infty} e^{-jwx} f'(x)\,dx = \underbrace{\left[e^{-jwx} f(x)\right]_{-\infty}^{\infty}}_{0} - (-jw) \int_{-\infty}^{\infty} e^{-jwx} f(x)\,dx$

$= jw(Ff)(w)$

since $f(x) \to 0$ as $x \to \pm\infty$

FOURIER TRANSFORM DISC.

1-dim F-trans. of $f: \mathbb{Z} \to \mathbb{C}$

$$\hat{f}(\omega) = (Ff)(\omega) = \sum_{-\infty}^{\infty} e^{-j\omega k} f(k)$$

1-dim F-inverse trans.

$$f(k) = (F^{-1}\hat{f})(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega k} \hat{f}(\omega)\, d\omega$$

EX. For f the PMF of $P_o(\lambda)$

$$f_{Po(\lambda)}(k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$\hat{f}(\omega) = \sum_{k=0}^{\infty} e^{-j\omega k} \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=0}^{\infty} (e^{-j\omega}\lambda)^k \frac{1}{k!} e^{-\lambda}$$

$$= e^{-\lambda} e^{e^{-j\omega}\lambda} = e^{\lambda(e^{-j\omega}-1)}$$

Taylor expansion

Properties

1. $(Ff(\sim - k_0))(\omega) = e^{-j\omega k_0} (Ff)(\omega)$

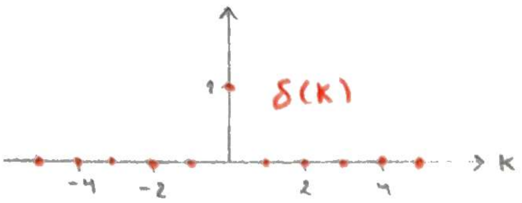
2. $(F(e^{j\omega} f))(\omega) = (Ff)(\omega - \omega_0)$

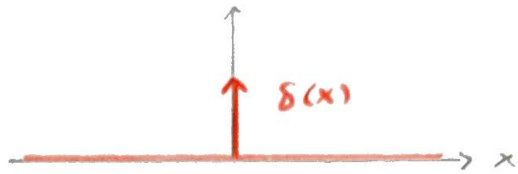
3. $(Ff(-x))(\omega) = (Ff)(-\omega)$

δ- FUNCTIONS

Discrete Kronecker δ-fct $\delta: \mathbb{Z} \to \{0, 1\}$ $\quad \delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$

Cont. Dirac δ-fct $\delta: \mathbb{R} \to [0, +\infty]$ $\quad \int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$





$(F\delta)(\omega) = 1$ for both disc & cont δ

Discrete: $(F\delta)(\omega) = \sum_{-\infty}^{\infty} e^{-j\omega k}\,\delta(k) = e^{-j\omega \cdot 0} = 1$

Continuous: $(F\delta)(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x}\,\delta(x)\,dx = e^{-j\omega\cdot 0} = 1$

All fct that grow at most polynomially has inverse transform
(but cannot nec. be calculated with formula, eg. 1)

Heavyside step fct: $\theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

has generalised derivative $\delta(x)$

proof: $\int_{-\infty}^{\infty} \theta'(x)\,f(x)\,dx$ want to prove this is $f(0)$

$= [\theta(x) f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \theta(x) f'(x)\,dx$ assume f has compact support (goes to 0 as $x \to \pm\infty$)

$= 0 - \int_0^{\infty} f'(x)\,dx$

$= -f(\text{"}\infty\text{"}) + f(0) = f(0)$ $\Rightarrow$ $\theta'(x)$ works as $\delta(x)$

CONVOLUTIONS (faltning)

Continuous convolution of $f, g : \mathbb{R} \to \mathbb{R}$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy = [z = x-y] = \int_{-\infty}^{\infty} f(z)\,g(x-z)\,dx$$
$$= (g * f)(x)$$

Discrete conv. of $f, g : \mathbb{Z} \to \mathbb{R}$

$$(f*g)(k) = \sum_{-\infty}^{\infty} f(k-l)\,g(l) = \ldots = \sum_{-\infty}^{\infty} f(\hat{l})\,g(k-\hat{l}) = (g*f)(k)$$

Thm

$$(F(f*g))(\omega) = (Ff)(\omega)(Fg)(\omega)$$

proof : $(F(f*g))(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x}\left(\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right)dx$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-j\omega x} g(y)}_{(Fg)(\omega)}\left(\underbrace{\int_{-\infty}^{\infty} e^{-j\omega(x-y)} f(x-y)\,dx}_{(Ff)(\omega)}\right)dy = (Ff)(\omega)\,(Fg)(\omega)$$

EX 1 continued.

$$f(x) = g(x) = f_{N(\mu,\sigma^2)}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$(f*g)(x)$?

$$(Ff)(\omega) = (Fg)(\omega) = e^{-j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$$

$$(F(f*g))(\omega) = e^{-2j\omega\mu - \omega^2\sigma^2} = e^{-j\omega(2\mu) - \frac{1}{2}\omega^2(2\sigma^2)}$$

$$= (F f_{N(2\mu, 2\sigma^2)})(\omega)$$

$$\Rightarrow \quad (f*g)(x) = f_{N(2\mu,2\sigma^2)}(x)$$

EX 3 continued.

$$f(k) = f_{Po(\lambda_1)}(k) \qquad g(k) = f_{Po(\lambda_2)}(k)$$

$$(F(f*g))(\omega) = (Ff)(\omega)\,(Fg)(\omega)$$

$$= e^{\lambda_1(e^{-j\omega}-1)}\, e^{\lambda_2(e^{-j\omega}-1)} = e^{(\lambda_1+\lambda_2)(e^{-j\omega}-1)}$$

$$= (F f_{Po(\lambda_1+\lambda_2)})(\omega)$$

$$\Rightarrow \quad (f*g)(k) = f_{Po(\lambda_1+\lambda_2)}(k)$$

Föreläsning 9

Today: HSU section 6.3C. PSD's

Consider a WSS process $X(t)$ with ACF $R_x(\tau) = E[X(t)X(t+\tau)]$

Def. Power Spectral Density

$$S_x(\omega) = (F R_x)(\omega) = \begin{cases} \int_{-\infty}^{\infty} e^{-j\omega\tau} R_x(\tau)\, d\tau & \text{cont} \\ \sum e^{-j\omega k} R_x(k) & \text{disc} \end{cases}$$

$$R_x(\tau) = (F^{-1} S_x)(\tau) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega\tau} S_x(\omega)\, d\omega & \text{cont} \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega\tau} S_x(\omega)\, d\omega & \text{disc} \end{cases}$$

Fourier transform

Ex. $X(t) = A \sin(\omega_0 t + \Phi)$ $\quad t \in \mathbb{R} \quad \omega_0 \in \mathbb{R}$ const

A, Φ indep r.v s.t $\Phi \sim unif[0, 2\pi]$ with ACF $R_x(\tau) = \frac{1}{2} E[A^2] \cos(\omega_0$

What is the PSD $S_x(\omega)$?

⋮

$$S_x(\omega) = \frac{\pi}{2} E[A^2] \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right)$$

Ex 2. ACF $R_x(\tau) = e^{-\alpha|\tau|}$ $\quad \tau \in \mathbb{R} \quad \alpha > 0$ const.

What is $S_x(\omega)$?

$$S_x(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} e^{-\alpha|\tau|}\, d\tau = \int_{0}^{\infty} e^{-(j\omega+\alpha)\tau}\, d\tau + \int_{-\infty}^{0} e^{-(j\omega-\alpha)\tau}\, d\tau$$

$$= \left[\frac{\exp(-(j\omega+\alpha)\tau)}{-(j\omega+\alpha)} \right]_0^{\infty} + \left[\frac{\exp(-(j\omega-\alpha)\tau)}{-(j\omega-\alpha)} \right]_{-\infty}^{0}$$

$$= \frac{1}{j\omega+\alpha} + \frac{1}{\alpha-j\omega} = \frac{2\alpha}{\alpha^2+\omega^2}$$

From this we immediately conclude that $R_x(\tau) = \frac{2\alpha}{\alpha^2+\tau^2}$ has PSD $S_x(\omega) = 2\pi e^{-\alpha|\omega|}$

Theorem

$S_x(\omega) \geq 0$, $S_x(\omega) = S_x(-\omega)$, $S_x(\omega) \in \mathbb{R}$

$$E[X(t)^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega \;/\; \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x(\omega)\, d\omega$$

proof : First statement very hard

Second : $S_x(-\omega) = \int_{-\infty}^{\infty} e^{-j(-\omega)\tau} R_x(\tau)\, d\tau = [\tau = -\hat{\tau}$

$$= \int_{-\infty}^{\infty} e^{-j\omega\hat{\tau}} R_x(-\hat{\tau})\, d\hat{\tau} = S_x(\omega)$$

Third : $\overline{S_x(\omega)} = \overline{\int_{-\infty}^{\infty} e^{-j\omega t} R_x(\tau)\, d\tau} = \int_{-\infty}^{\infty} e^{j\omega\tau} R_x(\tau)\, d\tau = S_x(\omega)$

Fourth : $E[X(t)^2] = R_x(0) = (F^{-1} S_x)(0)$

Recall two WSS processes $X(t)$ & $Y(t)$ are called jointly WSS if

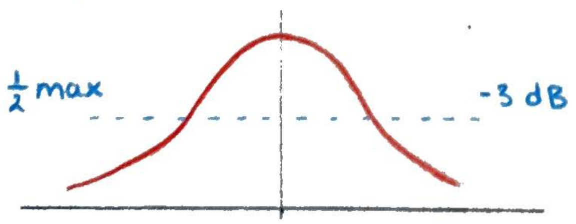
$R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)]$ depends on τ only

In that case we define the cross spectral density $S_{XY}(\omega) = (F R_{XY})(\omega)$

BAND WIDTH & WHITE NOISE

* The bandwithd of a WSS process $X(t)$ is a certain (not unique) measure of the width of the gap of $S_x(\omega)$

  E.g measured by selecting a -3 dB level



* White noise is a zero mean WSS (sometimes Gaussian) process $X(t)$ with PSD $S_x(\omega) = \sigma^2 > 0$ const

Therefore we must have $R_x(\tau) = \sigma^2 \delta(\tau)$

LINEAR TIME INVARIANT SYSTEMS

Def : An LTI system with insignal $x(t)$ and outsignal $y(t) = (Tx)(t)$ follows the two rules

1. $(T(\alpha x_1 + \beta x_2))(t) = \alpha (Tx_1)(t) + \beta (Tx_2)(t)$ linearity

2. $(T(x(\cdot - t_0)))(t) = (Tx)(t - t_0)$ doesn't change in time

Def : The impulse response of the LTI system is

$h(t) = (T\delta)(t)$

Theorem $(Tx)(t) = (h * x)(t)$

proof : Discrete time : $x(t) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(t-k)$

$$(Tx)(t) = \left(T\left(\sum_{k=-\infty}^{\infty} x(k)\, \delta(\cdot - k)\right)\right)(t) \overset{1.}{=} \sum_{k=-\infty}^{\infty} x(k)\, (T(\delta(\cdot - k)))(t)$$

$$\overset{2}{=} \sum_{-\infty}^{\infty} x(k) \underbrace{(T\delta)(t-k)}_{h(t-k)} = (h * x)(t)$$

impulse response

Henceforth we use a WSS process $X(t)$ with ACF $R_x(\tau)$ and mean μ_x as insignal to our LTI system

For the outsignal $Y(t) = (TX)(t) = (h * X)(t)$ we get

$$\mu_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(u) X(t-u)\, du\right] = \int_{-\infty}^{\infty} h(u) \underbrace{E[X(t-u)]}_{\mu_x(t-u) = \mu_x} du$$

$$= h * \mu_x = \mu_Y \quad \text{not dep. on } t$$

$$R_{XY}(t, t+\tau) = E[X(t) Y(t+\tau)] = E\left[X(t) \int_{-\infty}^{\infty} h(u) X(t+\tau-u)\, du\right]$$

$$= \int_{-\infty}^{\infty} h(u) R_x(\tau - u)\, du = (h * R_x)(\tau) = R_{XY}(\tau)$$

not dep on t

$$R_Y(t,t+\tau) = E\left[\int_{-\infty}^{\infty} h(u)\,X(t-u)\,du \int_{-\infty}^{\infty} h(v)\,X(t+\tau-v)\,dv\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)\,R_X(\tau-v+u)\,dv\,du = \int_{-\infty}^{\infty} h(u)\,(h * R_X)(\tau+u)\,du$$
$$= \int_{-\infty}^{\infty} h(-u)\,(h * R_X)(\tau-u)\,du = (h(-\cdot) * h * R_X)(\tau) = R_Y(\tau)$$

not dep on t

Def The frequency response (= transfer fct)

$H(\omega) = (Fh)(\omega)$ fourier transform

Theorem
$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$$
$$S_{XY}(\omega) = H(\omega)\,S_X(\omega)$$
$$S_{YX}(\omega) = \overline{H(\omega)}\,S_X(\omega)$$

proof $(F(h(-\cdot)))(\omega) = \overline{H(\omega)} \Rightarrow S_Y(\omega) = F(h(-\cdot) * h * R_X)(\omega)$
$$= \overline{H(\omega)}\,H(\omega)\,S_X(\omega)$$

2nd by inspection

3rd $S_{YX}(\omega) = (F R_{YX})(\omega) = (F R_{XY}(-\cdot))(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} R_{XY}(-\tau)\,d\tau$
$$= \int_{-\infty}^{\infty} e^{j\omega\tau} R_{XY}(\tau)\,d\tau = \overline{(F R_{XY})(\omega)} = \overline{H(\omega)\,S_X(\omega)}$$
$$= \overline{H(\omega)}\,S_X(\omega)$$

Föreläsning 10

6.1 - 6.5 in G&S

Hsu problem 6.29

An R(1) process $\{Y_n\}_{-\infty}^{\infty}$ is given by $Y_n = a Y_{n-1} + e_n$

where $\{e_n\}_{-\infty}^{\infty}$ is a discrete noise so that $S_e(\omega) = \sigma^2$

It can be considered an LTI system with insignal $\{e_n\}_{-\infty}^{\infty}$

and outsignal $\{Y_n\}_{-\infty}^{\infty}$

Find $S_Y(\omega)$!

Solution: $Y(\omega) = (\mathcal{F} Y_n)(\omega)$ & $e(\omega) = (\mathcal{F} e_n)(\omega)$ gives

$$Y(\omega) = (\mathcal{F} Y_n)(\omega) = (\mathcal{F}(h * e_n))(\omega) = (\mathcal{F} h)(\omega)(\mathcal{F} e_n)(\omega)$$

$$= H(\omega)\, e(\omega)$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} e^{-j\omega n} Y_n = \sum_{-\infty}^{\infty} e^{-j\omega n}(a Y_{n-1} + e_n) =$$

$$= e^{-j\omega} a \sum_{-\infty}^{\infty} e^{-j\omega(n-1)} Y_{n-1} + e(\omega) = e^{-j\omega} a\, Y(\omega) + e(\omega)$$

$$\Rightarrow \quad Y(\omega) = \frac{e(\omega)}{1 - a e^{-j\omega}} = H(\omega)\, e(\omega)$$

$$\Rightarrow \quad H(\omega) = \frac{1}{1 - a e^{-j\omega}}$$

$$\Rightarrow \quad S_Y(\omega) = |H(\omega)|^2 S_e(\omega) = \frac{\sigma^2}{|1 - a e^{-j\omega}|^2} = \frac{\sigma^2}{(1 - a e^{-j\omega})(1 - a e^{j\omega})}$$

$$= \frac{\sigma^2}{1 + a^2 - 2a\cos\omega}$$

Hsu sol: instead first calculate $R_Y(k) = \frac{\sigma^2}{1-a} a^{|n|}$

Fourier Transform $\Rightarrow$ $S_Y(\omega) = \sum_{-\infty}^{\infty} e^{-j\omega n} \frac{\sigma^2}{1-a} a^{|n|}$ = Same PSD

G&S book

* $\{X_n\}_{n\ge 0}^{\infty}$ time discrete Markov chain with values in the state space $S \subseteq \mathbb{Z}$ possesing the Markov property

* Markov property turns out to be equivalent with
$P(X_{n_{k+1}} = s_{k+1} \mid X_{n_k} = s_k, \dots, X_{n_0} = s_0) = P(X_{n_{k+1}} = s_{k+1} \mid X_{n_k} = s_k)$
for $0 \le n_0 < n_1 < \dots < n_k < n_{k+1}$ & $s_0, \dots, s_{k+1} \in S$

* We assume time homogeneity, ie $P(X_{n+1} = j \mid X_n = i) = p_{ij}$ does not depend on n

* We define transition matrix $(P)_{ij} = p_{ij}$
& n-step trans matrix $(P^{(n)})_{ij} = p_{ij}(n) = P(X_{m+n} = j \mid X_m = i)$
& have $P^{(n)} = P^n$

* Distribution at time n : $(\mu^{(n)})_j = P(X_n = j)$ giving $\mu^{n+m} = \mu^{(m)} P^n$

Ex. simple random walk

$$S = \mathbb{Z} \qquad p = \begin{cases} p & j = i+1 \\ q = 1-p & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_{jj}(n) = \begin{cases} \binom{n}{n/2} p^{n/2} q^{n/2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$p_{ij}(n) = \binom{n}{\frac{1}{2}(n+j-i)} p^{\frac{1}{2}(n+j-i)} q^{\frac{1}{2}(n-j+i)}$$

or 0

tot n.o steps : n
steps up - steps down : $j-i$

CLASSIFICATION OF STATES

$$f_{ij}(n) = \begin{cases} P(X_n = j,\ X_{n-1} \neq j, \dots, X_1 \neq j \mid X_0 = i) & n \geq 1 \\ 0 & n = 0 \end{cases}$$

$$f_{ij} = \sum_{n=0}^{\infty} f_{ij}(n) = \sum_{n=1}^{\infty} f_{ij}(n)$$

$$p_{ij}(0) = \delta_{ij} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

Note : i is recurrent iff $f_{ii} = 1$

Theorem

$$P_{ij}(s) = \sum_{n=0}^{\infty} s^n p_{ij}(n) \quad , \quad F_{ij}(s) = \sum_{n=0}^{\infty} s^n f_{ij}(n)$$

$$\Rightarrow \quad P_{ii}(s) = 1 + F_{ii}(s) P_{ii}(s) \quad , \quad P_{ij}(s) = F_{ij}(s) P_{jj}(s) \quad i \neq j$$

proof :

$$P_{ii}(s) = 1 + \sum_{n=1}^{\infty} s^n p_{ii}(n) = 1 + \sum_{n=1}^{\infty} s^n \sum_{k=1}^{n} f_{ii}(k) p_{ii}(n-k)$$

$$= 1 + \sum_{k=1}^{\infty} s^k f_{ii}(k) \sum_{n=k}^{\infty} s^{n-k} p_{ii}(n-k) = 1 + F_{ii}(s) P_{ii}(s)$$

$$P_{ij}(s) = \sum_{n=1}^{\infty} s^n p_{ij}(n) = \sum_{n=1}^{\infty} s^n \sum_{k=1}^{n} f_{ij}(k) p_{jj}(n-k)$$

$$= \underbrace{\sum_{k=1}^{\infty} s^k f_{ij}(k)}_{F_{ij}(s)} \underbrace{\sum_{n=k}^{\infty} s^{n-k} p_{jj}(n-k)}_{P_{jj}(s)} = F_{ij}(s) P_{jj}(s)$$

Theorem

1. j is recurrent iff $\sum_{n=0}^{\infty} p_{jj}(n) = +\infty$ and then $\sum_{n=0}^{\infty} p_{ij}(n) = +\infty$ for $i \neq j$ with $f_{ij} > 0$

2. j is transient iff $\sum_{n=0}^{\infty} p_{jj}(n) < +\infty$ and then $\sum_{n=0}^{\infty} p_{ij}(n) < +\infty$ for $i \neq j$ and $\lim_{n \to \infty} p_{ij}(n) = 0$

proof

$$F_{jj}(s) = \sum_{n=1}^{\infty} s^n f_{jj}(n) \nearrow \sum_{n=1}^{\infty} f_{jj}(n) = f_{jj} \quad \text{as} \quad s \nearrow 1$$

$$P_{jj}(s) = \sum_{n=0}^{\infty} s^n p_{jj}(n) \nearrow \sum_{n=0}^{\infty} p_{jj}(n) \quad \text{as} \quad s \nearrow 1$$

j recurrent $\iff f_{jj} = 1 \iff \lim_{s \nearrow 1} F_{jj}(s) = 1$

$$\iff \lim_{s\nearrow 1} \underbrace{\frac{1}{1 - F_{jj}(s)}}_{P_{jj}(s)} = +\infty \iff \lim_{s \nearrow 1} P_{jj}(s) = +\infty$$

$$\iff \sum_{n=0}^{\infty} P_{jj}(n) = +\infty$$

$i \neq j$: and then $\sum_{n=0}^{\infty} P_{ij}(n) = \lim_{s\nearrow 1} P_{ij}(s) = \lim_{s \nearrow 1} F_{ij}(s) P_{jj}(s)$

$$= f_{ij} \sum_{n=0}^{\infty} P_{jj}(n) \begin{cases} = +\infty & \text{for } f_{ij} > 0 \\ < +\infty & \text{for } \sum_{n=0}^{\infty} P_{jj}(n) < \infty \end{cases}$$

Def: $T_j = \min\{n \geq 1 : X_n = j\}$ the mean recurrence time μ_j of the state j is

$$\mu_j = E[T_j \mid X_0 = j] = \begin{cases} \sum_{n=1}^{\infty} n f_{jj}(n) & \text{for } f_{jj} = 1 \\ +\infty & \text{for } f_{jj} < 1 \end{cases}$$

Def: A recurrent / persistent state is called null if $\mu_j = +\infty$ and non-null otherwise.

Theorem

A recurrent state j is null iff $\lim_{n\to\infty} P_{jj}(n) = 0$

And then $\lim_{n\to\infty} P_{ij}(n) = 0$ for $i \neq j$

$$\iff \sum_{n=0}^{\infty} P_{ij}(n) = +\infty$$

Def: A state is called ergodic if it is non-null, recurrent & aperiodic

CLASSIFICATION OF CHAINS

Det: i communicates with j $(i \to j)$ if $p_{ij}(n) > 0$ for some $n \geq 0$

i & j intercommunicate $(i \leftrightarrow j)$ if $i \to j$ & $j \to i$

It is easy to see that $i \leftrightarrow j$ is an equivalent relation:
$(i \leftrightarrow i, \quad i \leftrightarrow j \Leftrightarrow j \leftrightarrow i, \quad i \leftrightarrow j \; j \leftrightarrow k \Rightarrow i \leftrightarrow k)$

Theorem

$i \leftrightarrow j \Rightarrow$
- i & j have same period
- i is transient iff j is transient
- i is non-null (recurrent) iff j is non-null (recurrent)

(part of) proof statement about transcience

For $i \leftrightarrow j$ we have $p_{ij}(m)\, p_{ji}(n) > 0$ for some $m, n \geq 0$

So $p_{ii}(m+n+r) \geq p_{ij}(m)\, p_{jj}(r)\, p_{ji}(r)$ ...

$$\Rightarrow \sum_{r=0}^{\infty} p_{jj}(r) \leq \sum_{r=0}^{\infty} \frac{p_{ii}(m+n+r)}{p_{ij}(m)\, p_{ji}(n)} < \infty \quad \text{if} \quad \sum_{r=0}^{\infty} p_{ii}(r) < \infty$$

So we have proved i trans. $\Rightarrow$ j trans.

Det: A set C of states is closed if

$i \in C, \; j \notin C \Rightarrow p_{ij}(n) = 0 \quad \forall n \qquad (\text{i.e } i \not\to j)$

A set of states C is irreducible if

$i \leftrightarrow j \quad \forall \; i, j \in C$

Theorem

$S = T \cup C_1 \cup C_2 \cup \ldots$ where T are transient states & $C_1, C_2, \ldots$ are closed irreducible sets of (recurrent) states proof: Let $C_1, C_2, \ldots$ be the equivalence classes for $\longleftrightarrow$ for the recurrent states

It remains to prove that the C_k's are closed

If $p_{ij} > 0$ for some $i \in C_k$ & $j \notin C_k$ so that $i \to j$

then we do not have $j \to i$ as this would mean $i \leftrightarrow j$ so that also $j \in C_k$

But this means that i is not recurrent, but transient

: Contradiction!

Theorem

If S is finite then at least one state is recurrent and all recurrent states are non-null

proof $1 = \sum_{j \in S} p_{ij}(n) \longrightarrow 0$ as $n \to \infty$ if all j are transient

(since all j trans. $\Rightarrow \lim_{n\to\infty} p_{ij}(n) = 0 \;\; \forall i, j$)

which is a contradiction

$\Rightarrow$ At least one state j recurrent

Now consider the closed set of all null-recurrent states C

Remember $S = T \cup C_1 \cup C_2 \cup \ldots$

If $C \neq \Phi$, so for $i \in C$

$1 = \sum_{j \in C} p_{ij}(n) \xrightarrow[n\to\infty]{} 0$ as we have learned that a recurrent state j is null iff $\lim_{n\to\infty} p_{jj}(n) = 0$

and then $\lim_{n\to\infty} p_{ij}(n) = 0 \;\; \forall i$

Contradiction!

STATIONARY DISTR. & LIMIT THEOREMS

π (a row matrix) is a stationary distribution for Markov Chain if

π is distribution & $\pi P = \pi$

Theorem

$\pi P^n = \pi$ & $\mu^{(m)} = \pi$ $\Rightarrow$ $\mu^{(m+n)} = \pi$ $n \geq 1$

Theorem

An irreducible chain has π iff all states are non-null recurrent

and in that case $\pi_j = 1/\mu_j$ $\forall j$

Theorem

For any aperiodic j $p_{jj}(n) \xrightarrow[n\to\infty]{} 1/\mu_j$

and $p_{ij}(n) \xrightarrow[n\to\infty]{} f_{ij}/\mu_j$ for $i \neq j$

Föreläsning 11

TIME REVERSIBILITY

In this section $\{X_n\}_{n=0}^{\infty}$ is irreducible non-null recurrent Markov chain being in steady state, eg $\mu^{(n)} = \pi$

We will consider the time reversed chain $Y_n = X_{N-n}$ for $n = 0, \dots, N$

Theorem

Y_n is a Markov chain with $P_{ij}^{Y} = \pi_j P_{ij}^{X} / \pi_i$

Def: X_n is time-reversible if $P_{ij}^{Y} = P_{ij}^{X}$

$$\Leftrightarrow \quad \frac{\pi_j P_{ji}^{X}}{\pi_i} = P_{ij}^{X}$$

proof: $P(Y_{n+1} = j \mid Y_n = i, Y_{n-1} = i_{n-1}, \dots, Y_0 = i_0)$

$$= \frac{P(X_{N-n-1} = j, X_{N-n} = i, X_{N-n+1} = i_{n-1}, \dots, X_N = i_0)}{P(X_{N-n} = i, X_{N-n+1} = i_{n-1}, \dots, X_N = i_0)}$$

$$= \frac{\pi_j P_{ji}^{X} P_{ii_{n-1}}^{X} \cdots P_{i_1 i_0}^{X}}{\pi_i P_{ii_{n-1}}^{X} \cdots P_{i_1 i_0}^{X}} = \frac{\pi_j P_{ji}^{X}}{\pi_i} = P_{ij}^{Y}$$

Theorem

If π is a distribution row matrix with $\pi_i P_{ij}^{X} = \pi_j P_{ji}^{X} \quad \forall\, i, j$

Then π is stationary distribution for X_n, & X_n is time reversible

proof: $(\pi P^{(X)})_j = \sum_k \pi_k P_{kj}^{X} = \sum_k \pi_j P_{jk}^{X} = \pi_j$

so π is stat. distr.

$$P_{ij}^{Y} = \frac{\pi_j P_{ji}^{X}}{\pi_i} = P_{ij}^{X} \quad \text{so } X_n \text{ time reversible}$$

Ex. Take $P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$ with $\alpha, \beta > 0$

Is chain time reversible?

Sol. Let's call the state space $S = \{0, 1\}$

Eq. in Thm amounts to $\pi_0 P_{01} = \pi_1 P_{10} \iff \pi_0 \alpha = \pi_1 \beta$

$\iff (\pi_0, \pi_1) = \left(\frac{\beta}{\alpha+\beta} \; \frac{\alpha}{\alpha+\beta} \right)$

This gives stat. distr. π as well as time reversibility

G & S Section 6.9

$\{X(t)\}_{t \geq 0}$ is a cont. time Markov chain

$P(X(t+s) = j \mid X(s) = i, X(s_{n-1}) = i_{n-1}, \ldots, X(s_0) = i_0)$

$= P(X(t+s) = j \mid X(s) = i) = P(X(t) = j \mid X(0) = i) = (P_t)_{ij}$

for $t+s > s > s_{n-1} > \ldots > s_0 \geq 0$

Theorem

$P_{s+t} = P_s P_t$ & $\mu^{(s+t)} = \mu^{(s)} P_t$

where $(\mu^{(t)})_i = P(X(t) = i)$

Def The generator $G = P_{0+}' = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(P_\varepsilon - I)$

Theorem

$P_t' = P_t G = G P_t$ & $P_t = e^{tG} = \sum_{k=0}^{\infty} \frac{1}{k!}(tG)^k$

Proof: $P_t' = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(P_{t+\varepsilon} - P_t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} P_t (P_\varepsilon - I) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(P_\varepsilon - I) P_t$

$= P_t G = G P_t$

proof. Second claim: Note that $\frac{d}{dt} e^{tG} = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{1}{k!} (tG)^k$

$$= \sum_{k=1}^{\infty} \frac{k}{k!} (tG)^{k-1} G = G e^{tG} = e^{tG} G$$

satisfies diff. eq. Therefor a sol.

Theorem

$$g_{ii} = (G)_{ii} \leq 0 \quad \& \quad g_{ij} = (G)_{ij} \geq 0 \quad \& \quad \sum_j g_{ij} = 0$$

proof: $G = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (P_\varepsilon - I)$ gives first two statements

$I_{ii} = 1$ P_ε is a prob: nonneg , $I_{ij} = 0$

while $\sum_j g_{ij} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sum_j ((P_\varepsilon)_{ij} - I_{ij}) = 0$

each row sum of transition matrix is 1

each row sum of identity matrix is 1

Theorem

The continuous time Markov chain stays an $\exp(-g_{ii})$ distributed time at state i, after which it switches value to j with prob. $\frac{g_{ij}}{-g_{ii}}$ for $j \neq i$

Note: $\sum_{j \neq i} g_{ij} = -g_{ii}$

Ex. Poisson process $S = \{0, 1, 2, \dots\} = \mathbb{N}$ state space

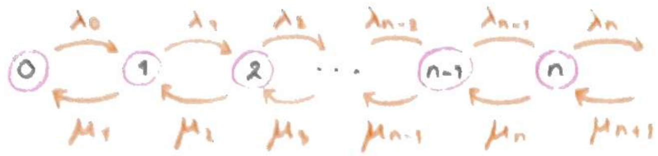
$\mu^{(0)} = (1 \ 0 \ 0 \ \dots)$ start..

$$G = \begin{bmatrix} -\lambda & \lambda & & 0 \\ & -\lambda & \lambda & \\ 0 & & -\lambda & \lambda \\ & & & \ddots \end{bmatrix}$$

Ex. Birth & death process $S = \{0, 1, 2, \dots\} = \mathbb{N}$

$\mu^{(0)}$ = what it is

$$G = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & & 0 \\ & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix}$$



STATIONARY DISTR. FOR CONT MARKOV CHAIN

Def. A stationary distribution π satisfies $\pi P_t = \pi$

Theorem

$\mu^{(s)} = \pi \Rightarrow \mu^{(s+t)} = \pi$ for $t > 0$

π is found by solving $\pi G = 0$

proof First claim proved as in disc. time

Second claim : $\pi P_t = \pi \iff \pi \sum_{k=0}^{\infty} \frac{1}{k!}(tG)^k = \pi$

$$\iff \pi \sum_{k=1}^{\infty} \frac{1}{k!}(tG)^k = 0 \iff \pi G = 0$$

Def Chain is irreducible if $p_{ij}(t) > 0$ for some $t > 0$ $\forall i,j$

Theorem

$p_{ij}(t) > 0$ for some $t > 0$ iff $p_{ij}(t) > 0$ $\forall t > 0$

Theorem

For an irreducible cont. time Markov chain either

1. π exists : $\pi_j = \lim_{t\to\infty} p_{ij}(t)$ $\forall i,j$ or
2. π [illegible] exist : $\lim p_{ij}(t) = 0$ $\forall i,j$

Föreläsning 12 EXERCISES

6.1.1) Show that any sequence of indep disc. r.v $\{X_n\}_{n=0}^{\infty}$ is a Markov chain. When is that chain time homogeneous?

Sol: $P\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0\}$

$$= \frac{P\{X_{n+1}=j, X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0\}}{P\{X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0\}}$$

$$= \frac{P(X_{n+1}=j)\, P(X_n=i)\, \cancel{P(X_{n-1}=i_{n-1}) \cdots P(X_0=i_0)}}{P(X_n=i) \cdots \cancel{P(X_0=i_0)}}$$

$$= \frac{P(X_{n+1}=j)\, P(X_n=i)}{P(X_n=i)} = P(X_{n+1}=j \mid X_n=i)$$

$$= P_{ij}(n, n+1)$$

Time homogeneity means no n-dependence for $P_{ij}(n,n+1)$ means $\{X_n\}_{n=0}^{\infty}$ IID

6.1.4 a) Let $\{X_n\}_{n=0}^{\infty}$ be a Markov chain and $\{n_r\}_{r=0}^{\infty}$ an increasing sequence of non-neg int.

Show that $Y_r = X_{n_r}$ is a Markov chain.

Find transition probabilities when $n_r = 2r$ & X_n is a simple random walk

Sol: $P(Y_{r+1}=i_{r+1} \mid Y_r = i_r, \ldots, Y_0 = i_0)$

$$= P(X_{n_{r+1}} = i_{r+1} \mid X_{n_r} = i_r, \ldots, X_{n_0} = i_0)$$

$$= P(X_{n_{r+1}} = i_{r+1} \mid X_{n_r} = i_r) = P(Y_{r+1} = i_{r+1} \mid Y_r = i_r)$$

2) $P(Y_{r+1} = j \mid Y_r = i) = P(X_{2(r+1)} = j \mid X_{2r} = i)$

$$= \begin{cases} p^2 & \text{for } j = i+2 \\ 2pq & \text{for } j = i \\ q^2 & \text{for } j = i-2 \end{cases}$$

6.1.2) A die is rolled repeatedly. Which of the following are Markov chains?

a) X_n = largest number shown up to n'th roll

b) N_n = n.o sixes in n rolls

c) C_n = time since the most recent 6 at time n

d) B_n = time to next six at time n

Sol. a) $P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$

$$= \begin{cases} 0 & \text{for } j < i \\ i/6 & \text{for } j = i \\ 1/6 & \text{for } j > i \end{cases}$$

b) $P(N_{n+1} = j \mid N_n = i, \ldots, N_0 = i_0) = \begin{cases} 5/6 & j < i \\ 1/6 & j = i \\ 0 & j > i \end{cases}$

c) $P(C_{n+1} = j \mid C_n = i, \ldots C_0 = i_0) = \begin{cases} 5/6 & j = i+1 \\ 1/6 & j = 0 \\ 0 & \text{otherwise} \end{cases}$

d) $P(B_{n+1} = j \mid B_n = i, \ldots, B_0 = i_0)$

$$B_{n+1} = \begin{cases} B_n - 1 & \text{for } B_n > 0 \\ Y & \text{for } B_n = 0 \end{cases}$$

Waiting time distr with param 1/6

$$= \begin{cases} 1 & \text{for } j = i-1 \geq 0 \\ (1/6)(5/6)^{j-1} & \text{for } i = 0, j > i \\ 0 & \text{otherwise} \end{cases}$$

So

6.1.10) Let X_n be a Markov chain. Show that

$$(*) = P(X_r = x_r \mid X_0 = x_0, \ldots, X_{r-1} = x_{r-1}, X_{r+1} = x_{r+1}, \ldots, X_n = x_n\}$$

$$= P(X_r = x_r \mid X_{r-1} = x_{r-1}, \ldots, X_{r+1} = x_{r+1}) = (*)$$

Sol. $(*) = \dfrac{\mu^{(0)}_{x_0} \, p_{x_0 x_1} \cdots p_{x_{n-1} x_n}}{\mu^{(0)}_{x_0} \, p_{x_0 x_1} \cdots p_{x_{r-2} x_{r-1}} \, p_{x_{r-1} x_{r+1}}(2) \, p_{x_{r+1} x_{r+2}} \cdots p_{x_{n-1} x_n}}$

$$= \frac{p_{x_{r-1} x_r} \, p_{x_r x_{r+1}} \, \mu^{(r-1)}_{x_{r-1}}}{p_{x_{r-1} x_{r+1}}^{(2)} \, \mu^{(r-1)}_{x_{r-1}}} = \frac{P(X_{r-1} = x_{r-1}, X_r = x_r, X_{r+1} = x_{r+1})}{P(X_{r-1} = x_{r-1}, X_{r+1} = x_{r+1})}$$

$$= (*)$$

6.1.12) A stochastic matrix P is called double stoch. if

$\sum_i p_{ij} = 1$, sub-stoch if $\sum_i p_{ij} < 1$

Show that if P is stoch. (double, sub) then P^n is too.

Sol. Claim for stoch. is clear.

Assume $\sum_i (P^n)_{ij} \substack{= 1 \\ \le 1}$ for $n = 1, \ldots, N$

$$\sum_i (P^{n+1})_{ij} = \sum_i \sum_k (P^N)_{ik} \, p_{kj} = \sum_k \Big(\sum_i (P^N)_{ik} \Big) p_{kj}$$

$$\substack{= \\ \le} \sum p_{kj} \substack{= \\ \le} 1$$

6.2.2) Let X be a Markov Chain containing an absorbing state S with which all other states communicate.

Show that all other states are trancient

Sol: $P(\text{no return to } i \mid X_0 = i) \geq P(X_{n_i} = s \mid X_0 = i) > 0$

for $n_i = \min\{n \geq 1 : p_{is}(n) \geq 0\}$

6.2.3) Show that a state is persistent iff the mean n.o visits to i having started at i is infinite

Sol. Let $I_K(\omega) = \begin{cases} 1 & \text{if } X_K(\omega) = i \\ 0, & \text{if } X_K(\omega) \neq i \end{cases}$

So that the n.o visits N to i is $N = \sum_{K=0}^{\infty} I_K$

It follows that $E(N \mid X_0 = i) = \sum_{K=0}^{\infty} P_{ii}(K) = \infty$

$\Longleftrightarrow$ i is recurrent

6.2.1) Last exit Let $l_{ij}(n) = P(X_n = j, X_{n-1} \neq i, \ldots, X_1 \neq i \mid X_0 = i)$

$L_{ij}(s) = \sum_{n=1}^{\infty} s^n l_{ij}(n)$

Show that $P_{ij}(s) = P_{ii}(s) L_{ij}(s)$ for $i \neq j$

Deduce that first passage times & last exit times have the same distribution for chains with $P_{ii}(s) = P_{jj}(s)$ $\forall i, j$

Give an example

Sol : $P_{ii}(s) = \sum_{n=0} s^n p_{ii}(n)$ will not depend on i for a simple random walk

Remember $P_{ij}(s) = F_{ij}(s) P_{jj}(s)$

for $i \neq j$ according to result in sect. 6.2

So if $P_{ii}(s) = P_{jj}(s) \quad \forall i,j$ then $P_{ii}(s) L_{ij}(s) = F_{ij}(s) P_{jj}(s)$

$\Rightarrow L_{ij}(s) = F_{ij}(s)$

$\Rightarrow l_{ij}(n) = f_{ij}(n) \quad \forall i,j,n$

$$P_{ij}(s) = \sum_{n=1}^{\infty} s^n p_{ij}(n) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} s^n p_{ii}(k)\, l_{ij}(n-k)$$

$$= \sum_{k=0}^{\infty} s^k p_{ii}(k) \sum_{n=k+1}^{\infty} s^{n-k} l_{ij}(n-k) = P_{ii}(s) L_{ij}(s)$$

6.3.2) Determine whether or not the random walk on the integers with $p_{i,i+2} = p$, $p_{i,i-1} = 1-p$, $p_{ij} = 0 \quad \forall j \neq i+2, i-1$ is persistent

Sol : Mean jump size : $2p - 1(1-p) = 3p-1$

It follows (?) that we have persistence iff $p = 1/3$

Alternatively $p_{ii}(n) = \begin{cases} p^k (1-p)^{2k} \binom{3k}{k} & \text{for } n = 3k \\ 0 & \text{otherwise} \end{cases}$

To judge whether $\sum_n p_{ii}(n) \lessgtr +\infty$:

$p_{ii}(3k) = p^k (1-p)^{2k} \dfrac{(3k)!}{k!\,(2k)!}$ use $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ for n large

$\Rightarrow = +\infty$ iff $p = 1/3$

6.4.4) Show by example that chains which are not irreducible may have many different stationary distributions

Sol. $S = \{0, 1\}$ & $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

gives that any distr. row matrix π is stationary distr.

6.3.3) Classify the states of the Markov chain with $P = \begin{bmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{bmatrix}$

Find the mean recurrence time of states

Sol. All states non-null, recurrent & aperiodic

$(\mu_0, \mu_1, \mu_2) = (1/\pi_0, 1/\pi_1, 1/\pi_2)$

where π is stationary distr. given by $\pi P = \pi$

$$\Leftrightarrow \begin{cases} \pi_0(1-2p) + \pi_1 p = \pi_0 \\ \pi_0 2p + \pi_1(1-2p) + \pi_2 2p = \pi_1 \\ p\pi_1 + (1-2p)\pi_2 = \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} \pi_1 = 2\pi_0 \\ \pi_1 = 2\pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$\Leftrightarrow (\pi_0, \pi_1, \pi_2) = (1/4, 1/2, 1/4)$$

6.3.4) A particle performs a discrete time random walk on the vertices of a cube.

At each step it remains where it is with prob. 1/4, or moves to one of its three neighbour vertices with prob 1/4 each

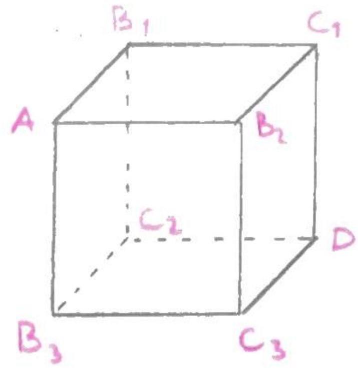
Let A & D denote two diametrically opposite vertices

If the walk starts at A, find

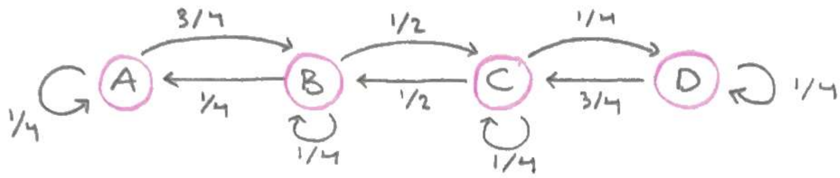
a) Mean n.o steps to first D-visit

b) —— " —— to first return to A

c) —— " —— visits to D before first return to A



Sol :



$$E(T_{AD}) = 1 + \tfrac{1}{4} E(T_{AD}) + \tfrac{3}{4} E(T_{BD})$$

$$E(T_{BD}) = 1 + \tfrac{1}{4} E(T_{AD}) + \tfrac{1}{4} E(T_{BD}) + \tfrac{1}{2} E(T_{CD})$$

$$E(T_{CD}) = 1 + \tfrac{1}{2} E(T_{BD}) + \tfrac{1}{4} E(T_{CD}) + \tfrac{1}{4} \cdot 0$$

system of eq $\Rightarrow$ $E(T_{AD}) = 40/3$

other two solved in similar way.

This is an IMPORTANT example...

6.4.6) Random walk on a graph

A particle performs a random walk on the vertices of a connected graph G which for simplicity we assume has neither loops nor multiple edges.

At each stage it moves to a neighbour vertice each with equal prob.

If G has $\eta < \infty$ edges, show that the stationary distr is given by $\pi_v = \frac{d_v}{2\eta}$ where d_v is the degree of vertice v

degree: n.o edges into v

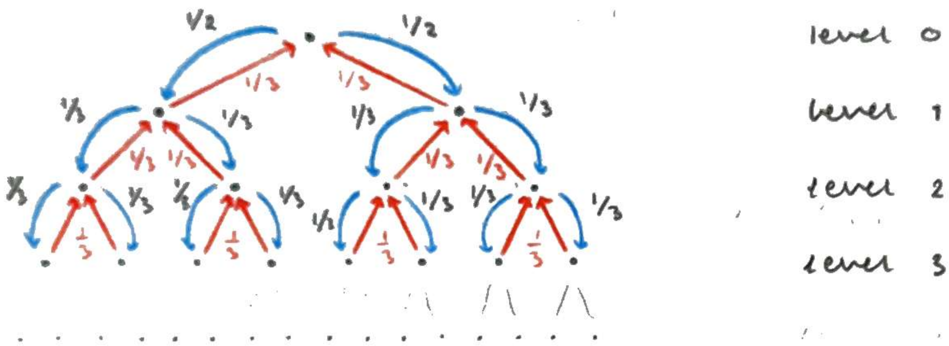
Sol: $\pi_v = c\, d_v$ obvious

$\sum_v d_v = 2\eta$ each road makes 2 connections

$\Rightarrow \pi_v = \dfrac{d_v}{2\eta}$

6.4.7) Show that a random walk on the infinite binary tree is trancient

Sol.



If X_n is level at time n we see that

$$p_{ij} = \begin{cases} 2/3 & \text{for } j = i+1 \\ 1/3 & \text{for } j = i-1 \end{cases} \quad (\text{for } i > 0)$$

Ie, simple random walk with $p = 2/3$ which is transient

6.4.8) At each time $n = 0, 1, 2, \ldots$ a number Y_n of particles enter a chamber, where $(Y_n)_{n=0}^{\infty}$ are indep $Po(\lambda)$-distributed.

Lifetimes of particles are geometrically distr. w param p.

Let X_n be n.o particles in chamber at time n

Show that X is Markov & find distribution

Sol. $X_{n+1} = \underbrace{\sum_{i=1} B_{i,n}}_{\text{how many survives}} + Y_n$ — how many new

where $\{B_{i,n}\}$ are indep with prop $P(B_{i,n}=0) = p$ &

$P(B_{i,n}=1) = 1-p$

gives Markov property

In equilibrium (steady state)

$$G_{n+1}(s) = E(s^{X_{n+1}}) = E(s^{Po(\lambda)})\, E\left(G_{Bin(1,1-p)}(s)^{X_n}\right)$$

probability generating function

$$= \left(\sum_{k=1}^{\infty} s^k \frac{\lambda^k}{k!} e^{-\lambda}\right)\left((p s^0 + (1-p) s^1)^{X_n}\right)$$

$$= e^{\lambda(s-1)}\, G_n(p + (1-p)s) = G_n(s)$$

$$\ldots \Rightarrow G_n(s) = e^{\lambda(s-1)p}$$

so $\pi_k = P(Po(\lambda/p) = k)$

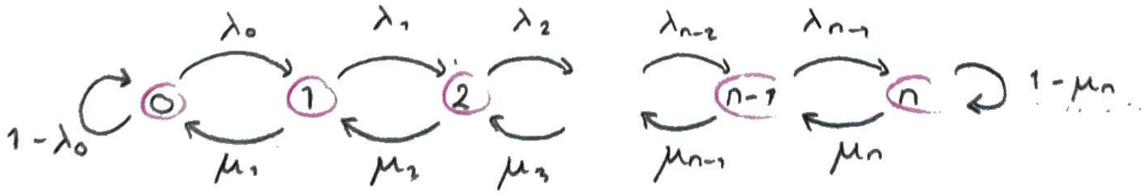
6.5.1) A random walk on the set $\{0, 1, \ldots, n-1, n\}$ has transition matrix $P_{00} = 1-\lambda_0$ $P_{bb} = 1-\mu_0$ $P_{i,i+1} = \lambda_i$ $P_{i+1,i} = \mu_{i+1}$

for $i = 0, \ldots, n-1$, $\mu_i, \lambda_i \in (0,1)$ $\lambda_i + \mu_i = 1$

Show that the chain is reversible in equilibrium

Sol. Time reversibility $\Leftrightarrow$ $\pi_j P_{ji} = \pi_i P_{ij}$

for some distribution row matrix π which must be stationary



$$\pi_k = \pi_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_n}{\mu_1 \mu_2 \cdots \mu_{n-1}} \qquad \pi_{k+1} P_{k+1,k} = \pi_k P_{k,k+1}$$

$$\pi_0 \frac{\lambda_0 \cdots \lambda_k}{\mu_1 \cdots \mu_{\cdot\cdot}} \mu_{k+1} = \pi_0 \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} \lambda_k$$

6.5.6 a) Is the following chain reversible ?

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \quad \text{with} \quad \alpha, \beta > 0$$

Sol : we have done this on lecture :

$S = \{0, 1\}$ state space $\quad \pi_0 P_{01} = \pi_1 P_{10} \iff \pi_0 \alpha = \pi_1 \beta$

$$\iff (\pi_0 \;\; \pi_1) = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$$

which gives reversibility

6.5.2 a) Kolmogorov criterion for reversibility

Let X be an irreducible non-null persistent aperiodic Markov chain. Show that X is reversible in equilibrium iff

$$(*) \quad P_{j_1 j_2} P_{j_2 j_3} \cdots P_{j_{n-1} j_n} P_{j_n j_1} = P_{j_1 j_n} P_{j_n j_{n-1}} \cdots P_{j_3 j_2} P_{j_2 j_1} \quad \forall n, j_1, \ldots, j_n$$

Sol : $\Rightarrow$ If chain reversible then multiply by π_{j_1}

$$\cancel{\pi_{j_1}} P_{j_1 j_2} P_{j_2 j_3} \cdots P_{j_{n-1} j_n} P_{j_n j_1} = P_{j_2 j_1} \pi_{j_2} P_{j_2 j_3} \cdots P_{j_n j_1}$$

$$= P_{j_2 j_1} P_{j_3 j_2} \pi_{j_3} \cdots P_{j_{n-1} j_n} P_{j_n j_1}$$

$$= \ldots$$

$$= P_{j_2 j_1} P_{j_3 j_2} \cdots P_{j_n j_{n-1}} P_{j_1 j_n} \cancel{\pi_{j_1}}$$

$\Leftarrow$ Assume $(*)$ holds. Consider $\pi_i p_{ij}$

$$\pi_i p_{ij} = \lim_{n\to\infty} p_{ji}(n) p_{ij} = p_{ij} \lim_{n\to\infty} \sum_{i_1, \ldots, i_n} p_{j i_1} \cdots p_{i_n i}$$

$$= p_{ji} \lim_{n\to\infty} \sum_{i_1, \ldots, i_n} p_{i i_1} \cdots p_{i_n j} = p_{ji} \lim_{n\to\infty} p_{ij}(n)$$

$$= p_{ji} \pi_j \qquad \text{so reversibility}$$

6.8.1) Superposition

Flies & wasps land on your dinner plate in the manner of indep poisson processes with resp. intensities λ & μ.

Show that the arrival of flying objects form a poisson process with intensity $\lambda+\mu$

Sol. Time between arrivals is $\min(\exp(\lambda), \exp(\mu))$

$= \exp(\lambda+\mu)$

After that arrival the competition starts over because of lack of memory

6.8.2) Thinning

insects land in the soup in the manner of a poisson process with intensity λ.

Each insect is green with prop. p indep of all other insects.

Show that the arrival process of green insects form a poisson process with intensity λp

Sol. As $\varphi_{\exp(\lambda)}(\omega) = E(e^{j\omega \exp(\lambda)}) = \dots = \frac{\lambda}{\lambda - j\omega}$
the time T between arrivals of green insects satisfies

$$\varphi_T(\omega) = E(e^{j\omega T}) = \sum_{k=1}^{\infty} p(1-p)^{k-1}\left(\frac{\lambda}{\lambda-j\omega}\right)^k$$

$$= p\frac{\lambda}{\lambda-j\omega}\sum_{k=1}^{\infty}(1-p)^{k-1}\left(\frac{\lambda}{\lambda-j\omega}\right)^k$$

$$= p\frac{\lambda}{\lambda-j\omega}\frac{1}{1-(1-p)\frac{\lambda}{\lambda-j\omega}} = \frac{\lambda p}{\lambda-j\omega-(1-p)\lambda} = \frac{\lambda p}{\lambda p - j\omega}$$

$$= \varphi_{\exp(p\lambda)}(\omega)$$

6.8.5) Let $B(t)$ be a process of simple birth with immigration $\lambda_n = n\lambda + \nu$ & $B(0) = 0$

Write down the sequence of differential difference equations for $p_n(t) = P(B(t) = n)$

Use them to show that $m(t) = E(B(t))$ satisfies $m'(t) = \lambda m(t) + \nu$

Sol. What is $m(t)$ when task has been done?

I.e. what is sol. to ODE with $m(0) = 0$

By inspection $m(t) = \nu(e^{\lambda t} - 1)$

$m'(t) - \lambda m(t) = \nu$

$$e^{-\lambda t} m'(t) - e^{-\lambda t}\lambda m(t) = e^{-\lambda t}\nu \quad \Rightarrow \quad \left| \nu L = \frac{d}{dt}\left(e^{-\lambda t} m(t)\right) \right|$$

$$e^{-\lambda t} m(t) = -\frac{1}{\lambda} e^{-\lambda t}\nu + C$$

$$m(t) = Ce^{\lambda t} - \frac{\nu}{\lambda} \qquad m(0) = 0 \Rightarrow C = \frac{\nu}{\lambda}$$

$$\Rightarrow \quad m(t) = \frac{\nu}{\lambda}\left(e^{\lambda t} - 1\right)$$

We have $P_t' = P_t G$ and $P_n(t) = \mu_n^{(t)}$ with $\mu^{(0)} = (1\ 0\ 0 \dots$

So that
$$P_n'(t) = (\mu^{(0)} P_t)'_n = (\mu^{(0)} P_t G)_n = (P_t G)_{0,n}$$
$$= \sum_{k=0}^{\infty} (P_t)_{0,k} G_{k,n} = (P_t)_{0,n} G_{n,n} + (P_t)_{0,n-1} G_{n-1,n}$$
$$= -p_n(t)(n\lambda + \nu) + p_{n-1}(t)((n-1)\lambda + \nu)$$

$$m'(t) = \frac{d}{dt} E[B(t)] = \frac{d}{dt}\sum_{n=1}^{\infty} n P(B(t) = n) = \sum_{n=1}^{\infty} n P_n'(t)$$
$$= \sum_{n=1}^{\infty} n\left(-p_n(t)(n\lambda + \nu) + p_{n-1}(t)((n-1)\lambda + \nu)\right)$$
$$= \sum_{n=1}^{\infty}\left((n-1)^2 p_{n-1}(t)\lambda - n^2 p_n(t)\lambda\right) + \sum_{n=1}^{\infty}\left(-\nu n\, p_n(t) + (n-1)\lambda p_{n-1}(t) + n\nu p_{n-1}(t)\right)$$
$$= (1-1)^2 p_{1-1}(t)\lambda + \sum_{n=1}^{\infty} \nu p_{n-1}(t) + \sum_{n=1}^{\infty} (n-1)\lambda p_{n-1}(t)$$
$$= 0 + \nu + \lambda m(t)$$

6.8.6) Let $N(t)$ be a birth process with intensities

$\lambda_0, \lambda_1, \lambda_2, \ldots$ and $N(0) = 0$

Show that $p_n(t) = P(N(t) = n)$ is given by

$$p_n(t) = \sum_{i=0}^{n} a_i e^{-\lambda_i t}$$

for some suitable $a_0, a_1, a_2, \ldots > 0$ when $\lambda_i \neq \lambda_j \quad \forall i \neq j$

Sol. $P(X(t) = n) = P\left(\sum_{i=1}^{n} \xi_i \le t \le \sum_{i=1}^{n+1} \xi_i\right) = \int_0^t f_{\sum_{i=1}^{n} \xi_i}(x) P(\xi_{n+1} > t-x) d$

ξ_i indep $\exp(\lambda_i)$

$$= \int_0^t (f_{\xi_1} * \cdots * f_{\xi_n})(x) \, P(\xi_{n+1} > t-x) \, dx$$

$$= \int_0^t (f_{\xi_1} * \cdots * f_{\xi_n})(x) \, e^{-\lambda_n (t-x)} \, dx \qquad \frac{1}{\lambda_n} f_{S_{n+1}}(t-x)$$

$$= (f_{\xi_1} * \cdots * f_{\xi_{n+1}})(x) \, \frac{1}{\lambda_n}$$

can find a_i

with CHF $\frac{1}{\lambda_n} \prod_{i=0}^{n} \frac{\lambda_i}{\lambda_i - jw} = \sum_{i=0}^{n} \frac{a_i \lambda_i}{\lambda_i - jw}$

So that $P(X(t) = n) = \sum_{i=0}^{n} a_i \lambda_i e^{-\lambda_i t}$ can incorporate λ_i into a

6.9.3) ... $M(\lambda)/M(\mu)/1$ queuing system with $X(0) = 0$

explain why Markov & find stationary distr when $\lambda < \mu$

Sol. $X(t)$ is a birth & death process with $\lambda_n = \lambda$ & $\mu_n = \mu$.

and therefore Markov.

stationary distr we found already in Ch. 9 of Hsu to be

$$\pi_n = p_n = \pi_0 \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \pi_0 \left(\frac{\lambda}{\mu}\right)^n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

one may also easily check that $\pi G = 0$ since

$$G = \begin{bmatrix} -\lambda & \lambda & & 0 \\ \mu & -(\lambda+\mu) & \lambda & \\ & \mu & -(\lambda+\mu) & \lambda \\ 0 & & \ddots & \ddots \end{bmatrix} \Rightarrow \lambda \pi_{n-1} - (\lambda+\mu)\pi_n + \mu \pi_{n+1} = 0$$

for π_n as above.

6.9.2) Show that for a Markov chain $(X(t))_{t\geq 0}$ with

$S = \{1,2\}$ and $G = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$ & $X(0)$ = whatever...

Find $P(X(t)=2 \mid X(0)=1, X(3t)=1)$ &

$P(X(t)=2 \mid X(0)=1, X(3t)=1, X(4t)=1)$

given $(P_t)_{ij} = p_{ij}(t)$ is known

Sol. By an earlier problem in G & S book where we established conditional independence of the past & the future, the above probabilities agree:

So equal to $\dfrac{P(X(t)=2, X(0)=1, X(3t)=1)}{P(X(0)=1, X(3t)=1)} = \dfrac{\mu_1^{(0)} p_{12}(t)\, p_{21}(t)}{\mu_1^{(0)} p_{11}(3t)}$

6.9.1) Consider MC $\{X(t)\}_{t\geq 0}$ with $S=\{1,2\}$ and $G = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$ $\lambda, \mu > 0$

a) Write down the forward equation & solve them to find $p_{ij}(t)$

c) Solve $\pi G = 0$ and check that it fits with $\lim_{t\to\infty} p_{ij}(t) = \pi_j$

Sol. we know $P_t = e^{tG} = \sum_{n\geq 0}^{\infty} t^n \frac{G^n}{n!} = [G = \Lambda^{-1} D \Lambda]$

$$= \Lambda^{-1} \sum_{n\geq 0}^{\infty} t^n \frac{D^n}{n!} \Lambda = \Lambda^{-1} e^{tD} \Lambda$$

we also know $P_t' = P_t G = G P_t$

$$\begin{bmatrix} p_{11}'(t) & p_{12}'(t) \\ p_{21}'(t) & p_{22}'(t) \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\mu p_{11}(t) + \lambda p_{12}(t) & \mu p_{11}(t) - \lambda p_{12}(t) \\ -\mu p_{21}(t) + \lambda p_{22}(t) & \mu p_{21}(t) - \lambda p_{22}(t) \end{bmatrix}$$ forward eq.

$$\left(= \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix} = \dots \quad \text{Backward eq.} \right)$$

It is enough to solve for diagonal elements of P_t since

$P_{12}(t) = 1 - P_{11}(t)$ & $P_{21}(t) = 1 - P_{22}(t)$

$$[\,p_{11}'(t) \quad p_{22}'(t)\,] = [\,-\mu p_{11}(t) + \lambda(1 - p_{11}(t)) \qquad -\lambda p_{22}(t) + \mu(1 - p_{22}(t))\,]$$

where $p_{12}(t) = 1 - p_{11}(t)$ & $p_{21}(t) = 1 - p_{22}(t)$

$$\Rightarrow \quad P_t = \begin{bmatrix} P_{11}(t) & P_{21}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix} = \frac{1}{\lambda+\mu} \begin{bmatrix} \lambda + \mu e^{-(\lambda+\mu)t} & \mu - \mu e^{-(\lambda+\mu)t} \\ \lambda - \lambda e^{-(\lambda+\mu)t} & \mu + \lambda e^{-(\lambda+\mu)t} \end{bmatrix}$$

Same ODE as in 6.8.5

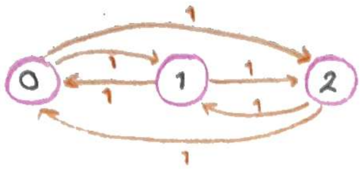
eigen values : 0 & $-(\lambda+\mu)$

$$\xrightarrow[t \to \infty]{} \begin{bmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\ \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix}$$

Exam Jan 2023 Task 6

$S = \{0, 1, 2\}$ $\quad \mu^{(0)} = (1\ 0\ 0)$ $\qquad G = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

Sol.



$T = \min\{t \geq 0 : X(t) = 2\}$

Char fct: $\Psi_T(\omega) = E(e^{j\omega T}) = E(e^{j\omega \exp(2)}) \left(\frac{1}{2} E(e^{j\omega T}) + \frac{1}{2} E(e^{j\omega 0})\right)$

exp 2 because diagonal of generator tells how long you are in, that state before change

char fct of exponential 2 fct

stays at 0

$= \frac{2}{2-j\omega}\left(\frac{1}{2}\Psi_T(\omega) + \frac{1}{2}\right)$

go to state 1 with prob $\frac{1}{2}$

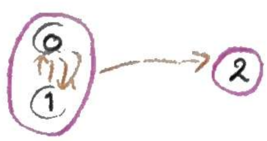
$\Rightarrow \left(1 - \frac{1}{2-j\omega}\right)\Psi_T(\omega) = \frac{1}{2-j\omega}$

$\Rightarrow \Psi_T(\omega) = \dfrac{\frac{1}{2-j\omega}}{1 - \frac{1}{2-j\omega}} = \dfrac{1}{2-j\omega-1} = \dfrac{1}{1-j\omega} = E(e^{j\omega \exp(1)})$

$\Rightarrow T \sim \exp(1)$

For state i we stay $\exp(-g_{ii})$ before switching to j with prob. $\frac{g_{ij}}{-g_{ii}}$ for $j \neq i$

Second sol:



$T = \sum_{i=1}^{N} \exp(2)$

$\Psi_T(\omega) = \sum_{n=1}^{\infty} \frac{1}{2}\cdot\left(1-\frac{1}{2}\right)^{n-1} \Psi_{\exp(2)}(\omega)^n$

n-1 jumps between 0 & 1
nth jump to 2

Waiting time distr param. 1/2

$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{2}{2-j\omega}\right)^n$

$= \dfrac{\frac{1}{2-j\omega}}{1-\frac{1}{2-j\omega}} = \dfrac{1}{1-j\omega}$

if p instead of $\frac{1}{2}$: $\ldots = \dfrac{p^2}{p^2 - j\omega} = \sum p(1-p)^{n-1}\Psi_{\exp(2)}(\omega)^n$

6.9.9) Let i be a transient state of a cont. time Markov chain X with $X(0) = i$

Show that the total time spent in i has an exp distr.

Sol. Couple of attempts to leave i

can come back if left, but eventually will leave & never come back

For each visit at i, we stay there an $\exp(-g_{ii})$ time

Then we leave i & come back to i eventually with prop f_{ii}

or instead never come back to i with prob $(1-f_{ii}) > 0$ because transient

This means tot time T_i spent in i:

$$\Psi_T(\omega) = E(e^{j\omega T_i}) = \sum_{n=1}^{\infty} (1-f_{ii}) f_{ii}^{n-1} \Psi_{\exp(-g_{ii})}(\omega)^n$$

waiting time distr.

$$= \sum_{n=1}^{\infty} (1-f_{ii}) f_{ii}^{n} \Big/ \left(\frac{-g_{ii}}{-g_{ii} - j\omega}\right)^n$$

geometric sum

$$= (1-f_{ii}) \left(\frac{-g_{ii}}{-g_{ii}-j\omega}\right) \Big/ \left(1 - \frac{-g_{ii} f_{ii}}{-g_{ii} - j\omega}\right) = \frac{(-g_{ii})(1-f_{ii})}{(-g_{ii} - j\omega) - (-g_{ii} f_{ii})}$$

$$= \frac{-g_{ii}(1-f_{ii})}{-g_{ii}(1-f_{ii}) - j\omega}$$

$$= \Psi_{\exp(-g_{ii}(1-f_{ii}))}(\omega)$$

6.9.10) Let X be an assymmetric simple random walk in cont. time on the non-neg int. with retention at 0 s.t

$$P_{ij}(h) = \begin{cases} \lambda h + o(h) & j = i+1 \quad i \geq 0 \\ \mu h + o(h) & j = i-1 \quad i \geq 1 \\ o(h) & \text{for other } j \neq i \\ 1 - (\lambda+\mu)h + o(h) & \text{for } j = i, \ i \geq 1 \\ 1 - \lambda h + o(h) & \text{for } j = i = 0 \end{cases}$$

Suppose $X(0) = 0$, $\lambda > \mu$

Show that the tot time V_r spent in state r is exponentially distr. with parameter $\lambda - \mu$

Sol. This is a birth- & death process. (starting at 0)

With $g_{ii} = \begin{cases} -(\lambda+\mu) & i \geq 1 \\ -\lambda & i=0 \end{cases}$ $\quad g_{ij} = \begin{cases} \lambda & j=i+1 \quad i \geq 0 \\ \mu & j=i-1 \quad i \geq 1 \end{cases}$

get by differentiating at 0

probability q_i of ever visiting 0 having started at i

satisfies $q_0 = 1 \qquad q_i = \frac{\mu}{\mu+\lambda} q_{i-1} + \frac{\lambda}{\mu+\lambda} q_{i+1} \qquad i \geq 1$

characteristic polynomial :

$q_i = \left(\frac{\mu}{\lambda}\right)^i = \underbrace{c_1 (\text{root } 1)^i}_{= \left(\frac{\mu}{\lambda}\right)^i} + \cancel{c_2 (\text{root } 2)^i}$ $\quad c_1 = 1 \quad$ $= 1$

has to go to 0 when $\longrightarrow \infty$

(= solving second order difference eq.)

Therefore the tot time V_0 spent in 0 is $\exp(\lambda(1-q_1))$

$$= \exp\left(\lambda\left(1-\frac{\mu}{\lambda}\right)\right) = \exp(\lambda-\mu)$$

Prob. p_i of ever returning to i having started there:

$$p_i = \frac{\mu}{\lambda+\mu} \cdot 1 + \frac{\lambda}{\lambda+\mu} \cdot q_1$$

$$= \frac{\mu}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} = \frac{2\mu}{\lambda+\mu}$$

if we go down : prob of coming back eventually is 1

if we go up ; prob is q_1

$\Rightarrow$ Tot time spent in $i \geq 1$ is $\exp\left((\lambda+\mu)\left(1-\frac{2\mu}{\lambda+\mu}\right)\right) = \exp(\lambda-\mu)$

6.11.1) Describe the jump chain for a birth & death process with rates λ_n & μ

Sol. Jump chain : sequence $\{Y_n\}_{n=0}^{\infty}$ of non-neg int. values that $\{X(t)\}_{t\geq 0}$ visits

Transition matrix :

$$P_{ij}^{(Y)} = \begin{cases} \frac{\lambda_i}{\lambda_i + \mu_i} & \text{for } j = i+1 \\ \frac{\mu_i}{\lambda_i + \mu_i} & \text{for } j = i-1 \geq 0 \end{cases}$$

6.11.2) Consider an immigration death process, ie a birth & death process with $\lambda_n = \lambda$ & $\mu_n = n\mu$

Find the transition matrix of the jump chain Z_n and show that it has stationary distr

$$\pi_n = \frac{1}{2n!}\left(1 + \frac{n}{\rho}\right)\rho^n e^{-\rho} \quad , \quad \rho = \frac{\lambda}{\mu}$$

Explain how this can differ from stationary for $X(t)$

Sol. $P_{i,i+1}(Z) = \frac{\lambda}{\lambda + i\mu}$ $\quad P_{i,i-1}(Z) = \frac{i\mu}{\lambda + i\mu}$

State distr to jump only cares for when we change state, (if we visit) not about how long we stay in each state

$$\pi_{i+1} P_{i+1,i}^{(Z)} + \pi_{i-1} P_{i-1,i}^{(Z)} = \pi_i \left(P_{i,i+1}^{(Z)} + P_{i,i-1}^{(Z)} \right) = \pi_i$$

with solution $\pi_i = \pi_0 \frac{P_{01} P_{12} \cdots P_{i-1,i}}{P_{10} P_{21} \cdots P_{i,i-1}}$

$$= \pi_0 \frac{1 \cdot \frac{\lambda}{\lambda+\mu} \cdot \cdots \cdot \frac{\lambda}{\lambda+(i-1)\mu}}{\frac{\mu}{\lambda+\mu} \cdot \cdots \cdot \frac{i\mu}{\lambda+i\mu}}$$

stationary distr

$$= \pi_0 \left(\frac{\lambda}{\mu}\right)^i \frac{1}{\lambda} \cdot \frac{1}{i!} (\lambda + i\mu)$$

$\Rightarrow$ claimed π_i

fix constants so that it sumizes to 1