Föreläsningsanteckningar

MVE170 - Basic Stochastic Processes

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Innehållsförteckning

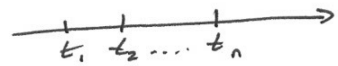
29/10 Stokastiske processer F2.

Def 2 of PP

$\{N(t), t \geq 0\}$ w/ $N(0)=0$ is a PP w/ rate λ if

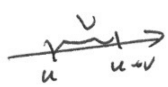
(A) It has independent increments on non-overlapping intervals

$N(t_{i+1}) - N(t_i) =$ the increment on $[t_i, t_{i+1}]$



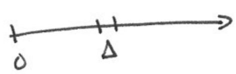
for $n \geq 3$ and any $t_1, \dots, t_n$ the r.v. $\{X(t_{i+1}) - X(t_i)\}_{i=1}^{n-1}$ are indep.

(B) It has stationary increments



The distr. of $N(u+v) - N(u)$ depends only on v (length of interval)

(C)



$P\{1 \text{ event in } \Delta\} \approx \lambda |\Delta|$

(D) $P\{\text{more than } 1 \text{ event in } \Delta\} = o(|\Delta|)$ $\quad \frac{o(|\Delta|)}{|\Delta|} \to 0$ when $|\Delta| \to 0$.

C, D may be replaced by C': $N(t) \sim \text{Poisson}(\lambda)$

Remarks

$N(u+v) - N(u)$ is distributed like $N(v) - N(0) = N(v) \sim \text{Poi}(\lambda v)$

1.1.2 Merging & splitting of the P.P.

① Y_1, Y_2 indep: $Y_i \sim \text{Poi}(\lambda_i)$, $Z = Y_1 + Y_2$ -

$$P\{Z=k\} = \sum_{j=0}^{k} P(\underbrace{Z}_{Y_1+Y_2}=k \mid Y_2=j) P(Y_2=j) = \sum_{j=0}^{k} P(Y_1 = k-j) P(Y_2=j) = \sum_{j=0}^{k} \frac{\lambda_1^{k-j}}{(k-j)!} e^{-\lambda_1} \frac{\lambda_2^j}{j!} e^{-\lambda_2}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda_1^{k-j} \lambda_2^j = \frac{e^{-(\lambda_1+\lambda_2)}}{k!} (\lambda_1+\lambda_2)^k \Rightarrow Z \sim \text{Poi}(\lambda_1+\lambda_2)$$

② X_1, X_2 indep: $X_i \sim \text{Exp}(\lambda_i)$

$Z = \min(X_1, X_2)$, distr. of Z?

$$P(Z>z) = P(X_1>z, X_2>z) \underset{indep}{=} P(X_1>z) P(X_2>z) = e^{-\lambda_1 z} e^{-\lambda_2 z} = e^{-(\lambda_1+\lambda_2)z}$$

$\cancel{\Rightarrow Z \sim \text{Exp}(\lambda_1+\lambda_2)}$ $P(Z \leq z) = 1 - e^{-(\lambda_1+\lambda_2)z} \Rightarrow Z \sim \text{Exp}(\lambda_1+\lambda_2)$

Merging of PPs

Thm 1.1.3 (a)

$\{N_i(t), t \geq 0\}$ independent PPs of rate λ_i for $i=1,2$. Consider for any $t \geq 0$

$N(t) = N_1(t) + N_2(t)$, then 1) $N(t)$ is a PP with rate $\lambda_1+\lambda_2$

2) Any arrival of $N(t)$ is with prop. $\frac{\lambda_i}{\lambda_1+\lambda_2}$ an arrival from process $N_i(t)$

Sketch of proof: 1)

- $N(t)$ is counting process w/ $N(0)=0$

It can be shown that

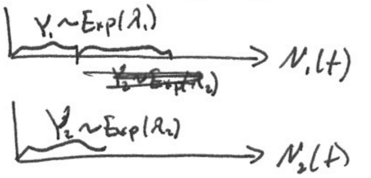
(A) $N(t)$ has independent increments on non-overlapping interv.

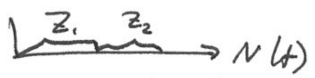
(B) $N(t)$ has stationary increments

(C') $N(t) = \underbrace{N_1(t)}_{Poi(\lambda_1)} + \underbrace{N_2(t)}_{Poi(\lambda_2)} \sim Poi(\lambda_1+\lambda_2)$

indep.

$\square$ by def 2.





$Z_1 = \min(Y_1, Y_2) \sim Exp(\lambda_1+\lambda_2)$

Proof of 2:

introduce the type variable $I_k = \begin{cases} 1 & \text{if } k\text{-th arrival is from } N_1(t) \\ 2 & \text{if } k\text{-th arrival is from } N_2(t) \end{cases}$

- Consider for $t \geq 0$, fixed k.

$$P(Z_k > t, I_k = 1) = \int_0^\infty P(Z_k > t, I_k = 1 \mid S_{k-1} = z) f_{S_{k-1}}(z)\,dz$$

$$P(Z_k > t, I_k = 1 \mid S_{k-1} = z)$$

{By the memoryless property of the PP the time from Z to the ~~first~~ first event from $N_i(t)$ is distr. like $Y_i \sim Exp(\lambda_i)$}

$$= P(Y_2 > Y_1 > t) = P\{\underbrace{Y_2 > Y_1}_{A}, Y_1 \in [t, \infty]\} = \int_t^\infty P(A \mid Y_1 = x) f_{Y_1}(x)\,dx$$

$$= \int_t^\infty P(Y_2 > Y_1 \mid Y_1 = x) f_{Y_1}(x)\,dx = \int_t^\infty P(Y_2 > x) f_{Y_1}(x)\,dx = \int_t^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x}\,dx = \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

ind of z.

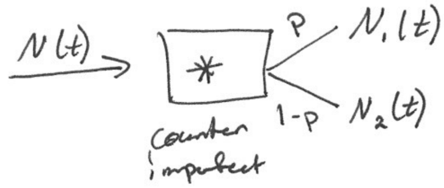
$$\Rightarrow P(Z_k > t, I_k = 1) = \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t} \underbrace{\int_{z=0}^\infty f_{S_{k-1}}(z)\,dz}_{=1} = \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

$$P(\underbrace{Z_k > 0}_{\text{certain event.}}, I_k = 1) = \frac{\lambda_1}{\lambda_1+\lambda_2} \qquad P(I_k = 1) = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

Moreover: $P\{Z_k > t, I_k = 1\} = P(Z_k > t)\,P(I_k = 1)$ for any $t \geq 0$, $k = 1, 2, \dots$

$\Rightarrow Z_k$ and I_k are independent

Splitting of the PP.



<u>Thm 1.1.3 b)</u>

$N(t)$ is PP of rate λ

each arrival is classified as type 1 w/ probability p_1 and independently of the other arrival, w/ type 2 w/ probability with prob. $p_2 = 1-p_1$

$N_i(t)$ is # of type i arrivals in $[0, t]$

Then $N_i(t)$ is a PP of parameter λp_i and $N_1(t), N_2(t)$ are independent

<u>Proof:</u>

It can be shown that PP is a PP by using def 2.

· Here we show that $N_i(t)$ is a $Poi(\lambda p_i t)$ and $N_1(t), N_2(t)$ are indep.

Consider prob.

$$P(N_1(t) = k, N(t) = k+n) = \cancel{P(N_1(t) = k, N_2(t) = n)}$$

$$= \underbrace{P(N_1(t) = k \mid N(t) = k+n)}_{Bin(n+k,\, p_1)} P(N(t) = k+n) = \binom{n+k}{k} p_1^k p_2^n \frac{(\lambda t)^{k+n}}{(k+n)!} e^{-\lambda t}$$

$$= \underbrace{\frac{(\lambda_1 p_1 t)^k}{k!} e^{-\lambda_1 p_1 t}}_{\sim Poi(\lambda p_1 t)} \cdot \underbrace{\frac{(\lambda_2 p_2 t)^n}{n!} e^{-\lambda_2 p_2 t}}_{\sim Poi(\lambda p_2 t)} = P(N_1(t) = k, N_2(t) = n)$$

$\Rightarrow$ independent N_1, N_2.

and $N_i(t)$ ~~is Poi of rate λp_i~~ $\sim Poi(\lambda p_i t)$

~~Queque~~ Quing theory. ~~Queuing~~ Queque

1.1.3 The $M|G|\infty$ Queque

- Customers arrive according to a ~~PP~~ PP of rate λ [M]
- There is an ample number of servers (infinitely many), s.t every arrival is immediately assigned server [∞]
- The service times are i.i.d w/ finite expectation μ [G] - general.

Let $L(t)$ - the # of occupied servers at time t.

It can be shown that $\lim_{t\to\infty} P(L(t) = k) = \frac{(\lambda\mu)^k}{k!} e^{-\lambda\mu}$ $k = 0, 1, 2, \ldots$

(only dependent on $\mu = \mathbb{E}[\text{service time}]$ not the distribution)

In a long run, the # of servers required is $L \sim Poi(\lambda\mu)$

L_0 - the # of servers in the system

$$\underbrace{P(L > L_0)}_{\text{probability that all servers are busy at customer arrival}} = P\Big(\underbrace{\frac{L - \lambda\mu}{\sqrt{\lambda\mu}}}_{\sim N(0,1)} > \frac{L_0 - \lambda\mu}{\sqrt{\lambda\mu}}\Big) \approx 1 - \phi\Big(\frac{L_0 - \lambda\mu}{\sqrt{\lambda\mu}}\Big)$$

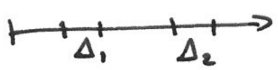
the service level

3/11 Stokastiska processer F3
Exercises.

1.1 Illegal parking 2 times/day for 1h each time. Checks occur according to a Poisson process w/ rate λ visits/h.

$P\{\text{getting a fine}\} = 1 - P\{\text{no fine}\} = 1 - P\{\text{no fine in first hour}\} P\{\text{no fine in second h}\}$

$\overset{\text{indep incr.}}{=} 1 - P\{N(1)=0\}^2 = 1 - e^{-2\lambda} = \{\lambda = \tfrac{1}{4}\} = .39$



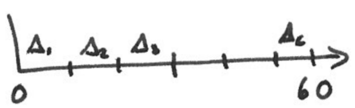
stat incr.

$\{\lambda = \tfrac{1}{2}\} = .62$
$\{\lambda = 1\} = .86$
$\{\lambda \geq 3\} = .99$

1.4 You arrive at bus stop at uniformly random time between 5 and 6 pm.
Bus #1 leaves every 10 min
Bus #2 leaves according to a Poisson Process w rate $\lambda = 6/\text{hour}$.

$P\{\text{take bus } \#1\} = ?$

Set time unit to minutes. Set origin at 5pm.

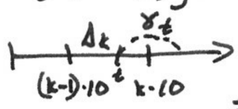


Rate of PP: $\lambda = \frac{6}{60} = \frac{1}{10}$ bus/min.
T - arrival time at busstop.
$T \sim U[0,60]$

$A = \{\text{you take bus } 1\}$

$* \; P\{A\} = \sum_{k=1}^{6} \underbrace{P\{A, T \in \Delta_k\}}_{\text{you arrive in } \Delta_k \text{ and take bus 1.}}$

$P\{A, T\in\Delta_k\} = \int_{\Delta_k} P\{A \mid T=t\} \underbrace{f_T(t)}_{\frac{1}{60}} dt = \frac{1}{60}\int_{\Delta_k} P\{\gamma_t > (10\cdot k - t)\} dt$ $\quad \gamma_t \sim Exp(\lambda)$



$= \frac{1}{60}\int_{(k-1)10}^{k\cdot 10} e^{-\frac{1}{10}(k\cdot 10 - t)} dt = \frac{1}{6}\left[1 - e^{-1}\right]$

By $*$: $P\{A\} = 1 - e^{-1} = 0.632 > \frac{1}{2}$

Why is $P\{A\} > \frac{1}{2}$?
Y_i - waiting time for bus i.

$E[Y] = \int_{-\infty}^{\infty} E[Y \mid T=t] f_T(t) dt$. $\quad E[Y_2] = \int_0^{60} \underbrace{E[Y_2 \mid T=t]}_{E[\gamma_t] = \frac{1}{\lambda} = 10} \cdot \frac{1}{60} dt = 10$

$E[Y_1] = \sum E[Y_1 \mid B_j] \cdot P\{B_j\} = \sum E[Y_1 \mid T\in\Delta_k] \cdot \frac{1}{6}$
$B_j = \{t \in \Delta_j\}$

$= \frac{1}{6}\sum E[k\cdot 10 - T \mid T\in\Delta_k] = \frac{1}{6}\sum\left(k\cdot 10 - \frac{1}{P(T\in\Delta_k)}\int_{\Delta_k} t \cdot f_T(t) dt\right)$

$$= \frac{1}{6}\sum\left(k\cdot 10 - \frac{1}{P(T\in\Delta_k)}\int_{\Delta_k} t\cdot f_T(t)\,dt\right) = \frac{1}{6}\sum\left(k\cdot 10 - \frac{k\cdot 10-(k-1)\cdot 10}{2}\right) = \frac{1}{6}\sum\left(k\cdot 10 - \frac{10}{2}\right)$$

$E[Y_1] = 5$

1.11

Customers = [orders for an oil tanker, arrive according to PP of rate 2/day]

Arrivals = each order is immediately assign a server (oil tanker is directed towards Amsterdam)

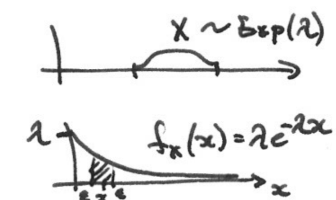
Service-Time = The average time is 10 days (sailing time)

What number of servers are busy at time t? [# of boats on the way to Amsterdam]

$L(t)$ = # of tankers on their way on day t.

t-large: $L(t) \sim Poi(2\cdot 10) \Rightarrow E[L(t)] \xrightarrow[t\to\infty]{} 20$

Clustering of PP

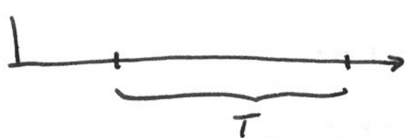


$P\{X\in(x-\epsilon, x+\epsilon)\} \approx 2\epsilon\cdot f_X(x)$

Small interarrival times are more probable than large interarrival times. Small interarrival times should occur more often.

Alm's approximation

- an interval of length T is given



- w some small fixed length $\qquad n$ - fixed

A: {there is in the first interval a subinterval of length w with n or more events}

- Alm's appr. gives $P(A)$

$$1 - P(n-1, \lambda w)\exp\left\{1 - \left(1 - \frac{\lambda w}{n}\right), \lambda(T-w)\right), p(n-1,\lambda w)\right\}$$

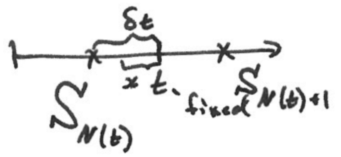
$$p(k, \lambda w) = e^{-\lambda w}\frac{(\lambda w)^k}{k!}$$

$$P(k, \lambda w) = \sum_{j=0}^{k} p(j, \lambda w)$$

Example p 10. Read this

1.6 a

$N(t)$ - PP of rate λ



δ_t - backward time from t to first event.

$$\delta_t = \begin{cases} t, & N(t)=0 \\ t - S_{N(t)} & \text{otherwise} \end{cases}$$

Find distrib. of δ_t.

$0 \le \delta_t \le t \qquad P\{\delta_t = t\} = P\{N(t)=0\} = e^{-\lambda t}$

$0 \le x < t$

$\cdot \{\delta_t > x\} = \{S_1 > t\} \cup \{S_1 < t-x, S_2 > t\} \cup \dots \cup \{S_k < t-x, S_{k+1} > t\}$

$\{\delta_t > x\} = \{S_1 > t\} \bigcup_{k=1}^{\infty} \{S_k < t-x, S_{k+1} > t\}$

all events involved are disjoint, thus $P\{\delta_t > x\} = \underbrace{P\{S_1 > t\}}_{e^{-\lambda t}} + \sum_{k=1}^{\infty} P\{S_k < t-x, S_{k+1} > t\}$

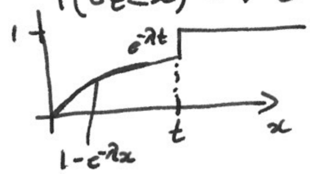
$P\{S_k < t-x, S_{k+1} > t\} = P\{N(t-x)=k, N(t) - N(t-x) = 0\}$ ind & stat. incr.

$= P(N(t-x)=k\} \cdot P\{N(x)=0\} = e^{-\lambda \cancel{x}} \frac{[\lambda(t-x)]^k}{k!} e^{-\lambda(t-x)} = e^{-\lambda t} \frac{[\lambda(t-x)]^k}{k!}$

$\Rightarrow P\{\delta_t > x\} = e^{-\lambda t}\left(1 + \sum_{k=1}^{\infty} \frac{[\lambda(t-x)]^k}{k!}\right)$

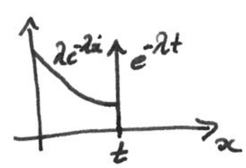
$= e^{-\lambda t} \cdot e^{\lambda(t-x)} = e^{\lambda x}$ for $x \in [0,t)$

$P(\delta_t \le x) = 1 - e^{-\lambda x}$



truncated exponential distribution

δ_t is a mixed r.v.



$\cancel{\delta_t(x)}$

$f_{\delta_t}(x) = [\lambda e^{-\lambda x} + \delta(x-t)]$ for $x \in [0,t]$.

δ_t, γ_t independent.

1.16

PP of rate 1 accident/day Given: $N(7)=7$ compute the cond. prob.

for 1 acc. ~~every~~ 1 day

$$P\{1 \text{ acc. every day} \mid N(7)=7\} = \frac{P(1 \text{ acc. every day}, N(7)=7)}{P\{N(7)=7\}} = \frac{\left(\frac{\lambda}{7} e^{-\frac{\lambda}{7}}\right)^7}{\frac{\lambda^7}{7!} e^{-\lambda}}$$

$$= \frac{\lambda^7 e^{-\frac{7\lambda}{7}} 7!}{7^7 \cdot \lambda^7 \cdot e^{-\lambda}} = \frac{7!}{7^7} = 0.0061$$

5/11 Stokastiske processer F4

Conditional distr. 2 exp.

X, Y jointly discrete

S_x, S_y ranges.

$p(x,y) = P\{X=x, Y=y\}$, $x \in S_x$, $y \in S_y$ | the joint mass function of X & Y

The marginal mass function

$$p_X(x) = P\{X=x\} = \sum_{y \in S_y} p(x,y)$$

take a fixed $y \in S_y$:

$$\underbrace{p_{X|Y}(x|y)}_{\text{conditional distribution of } x,\text{ given } Y=y} = P\{X=x \mid Y=y\} = \frac{p(x,y)}{p_Y(y)}$$

$$\{x \in S_x,\ p_{X|Y}(x|y)\}$$

Expected value:

$$E[X|Y=y] = \sum_{x \in S_x} x \cdot p_{X|Y}(x|y)$$

Define $E[X|Y]$ = to equal $E[X|Y=y]$ when $Y=y$. $E[X|Y]$ is a r.v is called the conditional expectation value of X given Y

$E[X|Y]$ has the following distribution

$$\{E[X|Y],\ y \in S_Y;\ p_Y(y)\}$$

$$E[E[X|Y]] = \sum_{y \in S_y} E[X|Y=y] \cdot p_Y(y) = \sum_{y \in S_y} \sum_{x \in S_x} x \cdot p_{X|Y}(x|y)\, p_Y(y)$$

$$= \sum_{x \in S_x} x \underbrace{\sum_{y \in S_y} \underbrace{p_{X|Y}(x|y)\, p_Y(y)}_{p(x,y)}}_{p_X(x)} = E[X]$$

The continuous case

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

TQ: Lemma 1.1.4

Let $\{N(t), t \geq 0\}$ be a PP w. rate λ and arrival times $S_1, S_2, \ldots$ Put $S_0 = 0$. for any $k \in [1, n]$ and any $t > 0$ we have

a) $$P\{S_k \leq x \mid N(t) = n\} = \sum_{j=k}^{n} \binom{n}{j} \left(\frac{x}{t}\right)^j \left(1 - \frac{x}{t}\right)^{n-j}$$

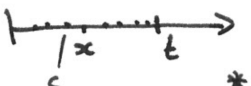
b) $$E[S_k \mid N(t) = n] = \frac{k \cdot t}{n+1}$$

$$E[S_k - S_{k-1} \mid N(t) = n] = \frac{t}{n+1}$$

Proof:

fix $n \geq 1$, k, t and x $\quad x \in [0, t]$

$N(t) = n$ in



a) $$* \; P\{S_k \leq x \mid N(t) = n\} = \frac{P\{S_k \leq x, N(t) = n\}}{P\{N(t) = n\}}$$

$$P\{S_k \leq x, N(t) = n\} = \sum_{j=k}^{n} P\{N(x) = j, N(t) - N(x) = n - j\} \stackrel{\text{ind incr.}}{=}$$

$$= \sum_{j=k}^{n} P\{N(x) = j\} \, P\{N(t) - N(x) = n - j\} \stackrel{\text{stat. incr.}}{=} \sum_{j=k}^{n} P\{N(x) = j\} \, P\{N(t - x) = n - j\}$$

$$= \sum_{j=k}^{n} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \frac{\lambda (t - x)^{n-j}}{(n-j)!} e^{-\lambda(t-x)} = \underbrace{e^{-\lambda t} \frac{(\lambda t)^n}{n!}}_{P\{N(t) = n\}} \sum_{j=k}^{n} \underbrace{\frac{n!}{j!(n-j)!}}_{\binom{n}{j}} \left(\frac{x}{t}\right)^j \underbrace{\left(\frac{t - x}{t}\right)^{n-j}}_{\left(1 - \frac{x}{t}\right)^{n-j}}$$

$$\Rightarrow * \; P\{S_k \leq x \mid N(t) = n\} = \sum_{j=k}^{n} \binom{n}{j} \left(\frac{x}{t}\right)^j \left(1 - \frac{x}{t}\right)^{n-j}$$

b) Use: if $Z \geq 0$ w/ $E[Z] < \infty$ then $E[Z] = \int_{-\infty}^{\infty} [1 - F_Z(z)] \, dz$ $\quad \cancel{P(Z}\; P\{Z = z\}$

$$E[S_k \mid N(t) = n] = \int_0^{\infty} [1 - P\{S_k \leq x \mid N(t) = n\}] \, dx$$

$$= \int_0^t \Big[\underbrace{1 - \sum_{j=k}^{n} \binom{n}{j} \left(\frac{x}{t}\right)^j \left(1 - \frac{x}{t}\right)^{n-j}}_{\text{by bin thm. } \sum_{j=0}^{k-1} \binom{n}{j} \left(\frac{x}{t}\right)^j \left(1 - \frac{x}{t}\right)^{n-j}}\Big] dx = \sum_{j=0}^{k-1} \binom{n}{j} \int_0^t \left(\frac{x}{t}\right)^j \left(1 - \frac{x}{t}\right)^{n-j} dx$$

$$= \left\{u = \frac{x}{t}\right\} = \sum_{j=0}^{k-1} \binom{n}{j} t \underbrace{\int_0^1 u^j (1-u)^{n-j} \, du}_{B(j+1,\, n-j+1)} = \sum_{j=0}^{k-1} \binom{n}{j} t \frac{j! \, (n-j)!}{(n+1-j+j)!} = \frac{t}{n+1} \sum_{j=0}^{k-1} 1 = \frac{k \cdot t}{n+1}$$

$$\left[\begin{array}{ll} B(x,y) = \int_0^1 u^{x-1} (1-u)^{y-1} \, du & x > 0, y > 0 \\ x, y \in \mathbb{Z}^+ \quad B(x,y) = \dfrac{(x-1)! \, (y-1)!}{(x+y-1)!} & \end{array}\right]$$

$$E[S_k - S_{k-1} \mid N(t) = n] = E[S_k \mid N(t) = n] - E[S_{k-1} \mid N(t) = n] = \frac{k \cdot t}{n+1} - \frac{(k-1) t}{n+1} = \frac{t}{n+1}$$

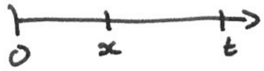
1.1.4 The PP & uniform distribution.

$X_1, \ldots, X_n$ iid $X_i \sim U[0,t]$

order: $X_{(1)} = \min(X_1, \ldots, X_n)$
$X_{(2)}$ = next in size after $X_{(1)}$
$X_{(n)}$ - largest.

} The order statistic of $X_1 \ldots X_n$.

$$P\{X_{(k)} \le x\} = \sum_{j=k}^{n} P\{j \text{ points in } [0,x], (n-j) \text{ points in } [x,t]\}$$



$$= \sum_{j=k}^{n} \binom{n}{j} \left(\frac{x}{t}\right)^j \left(1 - \frac{x}{t}\right)^{n-j} = P\{S_k \le x \mid N(t) = n\}$$

AND:

$$P\{S_1 \le x_1, S_2 \le x_2, \ldots, S_n \le x_n \mid N(t) = n\} = P\{X_{(1)} \le x_1, X_{(2)} \le x_2, \ldots, X_{(n)} \le x_n\}$$

Read about this in book. ook.

Mean & variance of random sums

- $X_1, X_2, \ldots$ iid $E[X_i] = \mu$
- N - a r.v. w/ values $0, 1, 2, \ldots$ and $E[N] = \nu$ N indep of all X_i
- $S = \sum_{k=1}^{N} X_k$

$E[S] = E[E[S|N]]$

$E[S|N=n] = E[\sum_1^N S_k \mid N=n] = E[\sum_1^n X_i \mid N=n] = \sum_{i=1}^{n} E[X_i] = n \cdot \mu$

$E[S|N] = N \cdot \mu$

$E[S] = E[N \cdot \mu] = \mu \cdot \nu$

$E[S] = E[X_1] E[N]$ Wald's equation, true if $\{N=n\}$ and $X_{n+1}, \ldots$ are indep. $\forall n \ge 1$.

$Var(S) = Var(X_i) E[N] + Var(N) E[X_i]^2$

Ex: p. 432 A.1

U - unloading time for boat $U \sim f(x)$ $E[U] = \gamma < \infty$

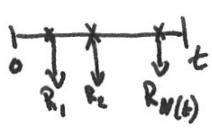
$R_1, R_2, \ldots$ interruption times

$N(t)$ - the PP formed by the interruptions

All r.v. involved indep.

C = total unloading time, find $E[C]$.

$E[C] = E[E[C|U]]$, $E[C|U] = E[t + \sum_1^{N(t)} R_j] = t + E[N(t)] \cdot E[R_j] = t + \lambda t \cdot E[R_i]$

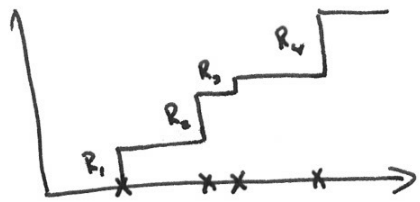


$E[C|U] = U(1 + \lambda E[R_i])$

$E[C] = \gamma(1 + \lambda E[R_i])$

in ex. Let $X(t) = R_1 + \dots + R_{N(t)}$ - total interruption time up to time t.

$\{X(t), t \geq 0\}$ is a compound P.P.



Counts number of demands up to time t.

Def. 1.2 Compound PP

- $\{N(t), t \geq 0\}$ of rate λ
- $D_1, D_2, \dots$ iid and indep of $N(t)$

$$X(t) = \sum_{i=1}^{N(t)} D_i$$

$\Rightarrow \{X(t), t \geq 0\}$ is a compound PP.

When D_i is discrete w/ values $0, 1, \dots$ $X(t)$ is the disc. comp. PP.

Thm 1.2.1 $X(t)$ discrete

b) $r_j(t) = P\{X(t) = j\}$ $j = 0, 1, \dots$

Then $r_j(t) = \frac{\lambda t}{j} \sum_{k=0}^{j-1} (j-k)\, a_{j-k}\, r_k(t)$ $\quad r_0(t) = e^{-\lambda t(1-a_0)}$

$a_j = P\{D_1 = j\}$ $j = 0, 1, \dots$

Read part a.
+ about generating functions.
Will be theoretical question

6/11 Stokastiske F

Ch2 ~~Renewal~~ reward processes

$\{X_i\}_{i=1}^{\infty}$ iid $E[X_i] < \infty$ $\mu_1 < \infty$
non-negative

Assume events occur in time w/ interoccurence times $X_1, X_2, \dots$

Def. 2.1.1
The counting process $\{N(t), t \geq 0\}$ with $N(0) = 0$ & $N(t) = \{\max n : S_n \leq$
is called a renewal process generated by $X_1, X_2, \dots$ $\{\max n : S_n \leq t$

- When $X_i \sim Exp(\lambda)$, $N(t)$ is the PP.
- Can the # of renewals up to t be infinite?

$S_n = X_1 + \dots + X_n \leftarrow$ iid w $E[X_i] = \mu_i$

By the strong Law of Large Numbers

$P\{\lim \frac{S_n}{n} = \mu_1\} = 1$

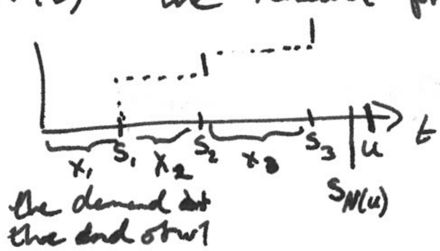
for large n, $S_n \approx \mu_1 \cdot n$, Take $n > \frac{t}{\mu_1} \Rightarrow S_n > t$

The number of renewals up to t is finite w/ prob. 1.

Ex 2.1.2

An inventory problem

- The demands for a single product in weeks 1, 2, ... are continuous ~~random~~ r.v. iid r.v $X_1, X_2, \dots$
- $N(t)$ – the renewal process generated by $X_1, X_2, \dots$



S_n = cumulative stock which is out in the end of week n.

u – initial stock

$N(u)+1$ week # when inv. has run out of stock

2.1.1 The renewal function

- $\{N(t), t \geq 0\}$ – a renewal p. generated by $X_1, X_2, \dots$
- The renewal function $M(t) = E[N(t)]$
- $S_1, S_2, \dots$ renewal times.

$F_n(t) = P\{S_n \leq t, t \geq 0\}$ $F_1(t) = F(t)$

$\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\}$

TQ Lemma 2.4.1

For any $t>0$: $M(t) = \sum_{n=1}^{\infty} F_n(t)$

Proof:

$N(t)$ is a non-neg. integer valued r.v

(Ap A p434) $E[N(t)] = \sum_{k=0}^{\infty} P\{N(t)>k\} \underset{n=k+1}{=} \sum_{n=1}^{\infty} P\{N(t)>n-1\}$

$= \sum_{n=1}^{\infty} P\{N(t) \geq n\} = \sum_{n=1}^{\infty} F_n(t)$ □

Remarks

- $M(t)$ is finite for every t.
- If $X_i \sim f(x)$ then $M(t)$ satisfies $M(t) = F(t) + \int_0^t M(t-x) f(x)\,dx$

Computation of $M(t)$ when $X_i \sim \text{Erlang}(r, \lambda)$

$Y_1, \dots, Y_m$ i.i.d $\sim \text{Exp}(\lambda)$ $\quad Z = \sum_{i=1}^{m} Y_i \sim \text{Erlang}(m, \lambda)$

- Z_1, Z_2, Z_i ind, $Z_i \sim \text{Erlang}(m_i, \lambda)$

$Z = Z_1 + Z_2 + \dots \sim \text{Erlang}(m_1 + \dots + m_n, \lambda)$

- $\{N(t), t \geq 0\}$, $X_i \sim \text{Erlang}(r, \lambda)$ $\quad S_n = X_1 + \dots + X_n \sim \text{Erlang}(r \cdot n, \lambda)$

$P\{S_n \leq t\}$, S_n is distr. Z_{nr}, the nr-th arrival time of a PP of param λ.

$P\{S_n \leq t\} = P\{Z_{nr} \leq t\} = P\{\tilde{N}(t) \geq nr\}$

$= 1 - \sum_{k=0}^{nr-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

Thus $\boxed{M(t) = \sum_{n=1}^{\infty} F_n(t) = \sum_{n=1}^{\infty} \left[1 - \sum_{k=0}^{nr-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}\right]}$

For $r=1$; $M(t) = \lambda t$

$P\{N(t)=n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\} = P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$

$= \sum_{k=nr}^{(n+1)r-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

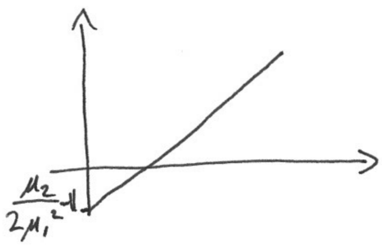
Assymptotic expansion of $M(t)$, $\{N(t), t \geq 0\}$ a renewal process

$X_1, X_2, \ldots,$ - inter occ. times

$f_{X_i}(x) > 0$ on some interval.

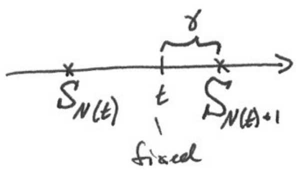
$E[X_1] = \mu_1 \quad E[X_2] = \mu_2$

Then $M(t) = E[N(t)] \approx \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1$ for large t.



2.1.2 The excess variable

$\{N(t), t \geq 0\}$ a renewal process, $E[X_1] = \mu_1$



γt - the time from t to the first event to come.

$\boxed{TQ}$ Lemma 2.1.2

For any $t > 0$ $\quad E[\gamma t] = \mu_1[1 + M(t)] - t$

Proof:

$\gamma t = S_{N(t)+1} - t \qquad S_{N(t)+1} = \sum_{k=1}^{N(t)+1} X_k$

$\{N(t)+1 = n\} = \{N(t) = n-1\} = \{S_{n-1} \leq t, S_n > t\}$

$S_{n-1} = \sum_{i=1}^{n-1} X_i \qquad S_n = \sum_{i=1}^{n} X_i$ — indep of $X_{n+1}, X_{n+2}, \ldots$

By Wald's equation

$$E\left[\sum_{k=1}^{N(t)+1} X_k\right] = E[N(t)+1] \cdot E[X_1] = [M(t)+1]\mu_n$$

$\Rightarrow E[\gamma t] = \mu_1[M(t)+1] - t \quad$ where $\mu_k = E[X_1^k]$

Approximation of $E[\gamma_t]$, in case X is con.

$$E[\gamma t] \xrightarrow[t \to \infty]{} \frac{\mu_2}{2\mu_1} \qquad E[\gamma t^2] \xrightarrow[t \to \infty]{} \frac{\mu_3}{3\mu_2}$$

Ex 2.13

An (S,s) inventory system, w/ certain products in stock.

· $X_1, X_2, \dots$ i.i.d r.v. $X_i \sim f(x)$ $E[X_i]=\alpha$, $E[X_i^2]=\sigma^2$

X_i – demand of stock in week $i=1,2,\dots$

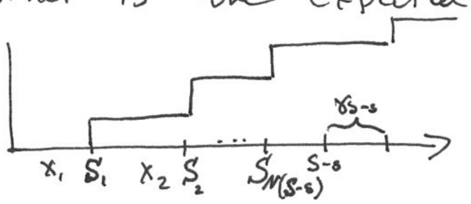
Any demand exceeding the current inventory is backlogged until inventory becomes available (by arrival of next order).

The inventory position is renewed in the beginning of each week. It is controlled by an (S,s) rule. X – the inventory level by the renewal.

$X<s$ replenishment is ordered of size $S-X$

$X \geq s$ no order is placed. $s<S$.

What is the expected value of the order size?



$N(t)$ renewal process generated by $X_1, \dots$

S_n – the amount of stock that is out ~~at~~ the end of week n.

$S-S_n$ = the amount left ~~at~~ the end of week n.

In week $N(S-s)+2$ replenishment is ordered of size $S-s+\gamma_{S-s}$

γ_{S-s} = skillnaden mellan $S-s$ ~~och~~ $X_{N(S-s)+2}$

$$E[S-s+\gamma_{S-s}] = S-s+E[\gamma_{S-s}]$$

$$E[\gamma_{S-s}] \approx \frac{\mu_2}{2\mu_1} = \frac{\alpha^2+\sigma^2}{2\alpha}.$$

10/11 Stokastiske processer

Pr. 1.12

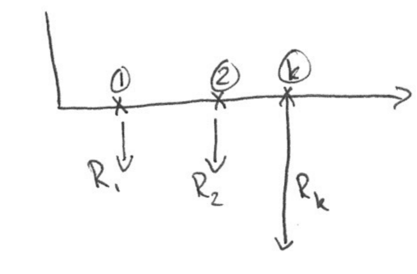
Customers arrive according to a PP of rate λ.

R - repair time of an item. $R \sim U[a,b]$

Upon arrival the exact repair time can be determined, if $R \geq \frac{a+b}{2}$ the customer gets a loaner item

The supply of loaners is sufficient

In the long run, what is the average number of loaners lent?



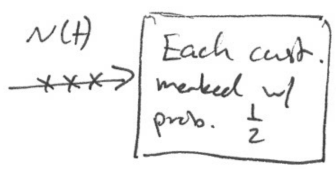
$R_1, R_2, \ldots$ repair times of cust. i.

$$P\{\text{cust. } k \text{ gets a loaner}\} = P\{R_k > \frac{a+b}{2}\}$$

$R \sim U[a,b]$ $\quad P\{R \geq x\} = \frac{x-a}{b-a}$, $f_R(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$

$$E[R] = \frac{a+b}{2}$$

$$\Rightarrow P\{R_k > \frac{a+b}{2}\} = \frac{1}{2}$$



$\to N_1(t)$ w/ loaner $\sim$ P.P. of rate $\frac{\lambda}{2}$

$\to N_2(t)$ w/o loaner.

Customers arrive according to $N_1(t)$, each cust. gets a server (loaner)

$$E[\text{service time}] = E[R \mid R > \frac{a+b}{2}] = \frac{1}{P\{R > \frac{a+b}{2}\}} \int_{\{x:\, x \in [\frac{a+b}{2}, b]\}} r f_R(r)\, dr$$

$$= \frac{1}{\frac{1}{2}} \int_{\frac{a+b}{2}}^{b} x \cdot \frac{1}{b-a} dx = \frac{a+3b}{4}$$

M, G, ∞ model.

$L(t)$ = the # of loaner out when $t \to$ large

$$L(t) \sim Poi\left(\frac{\lambda}{2} \cdot \frac{a+3b}{2}\right)$$

$$E[L(t)] = \frac{\lambda(a+3b)}{4}$$

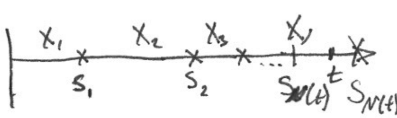
Pr. 2.1

. The lifetime of a street lamp is a r.v w/ distribution Erlang$(2, \mu)$

The lamp is replaced by a new one upon failure or when scheduled at times $T, 2T, \ldots$ all lifetimes are independent

What is the expected # of street lamps used in a scheduled interval?

X_i are iid $\sim$ Erlang$(2, \mu)$



Let $N(t)$ be the renewal process generated by $X_1, X_2, \ldots$

The # of lamps used in $[0,T] = N(T)+1$, $E[N(T)+1] = E[N(T)]+1 = M(T)+1$

P. 36 [24] $X_i \sim Er(r, \lambda) \Rightarrow M(t) = \sum_{n=1}^{\infty} \left[1 - \sum_{k=0}^{r \cdot n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}\right]$

$$M(T)+1 = \sum_{n=1}\left[1-\sum_{k=0}^{\dots}\frac{(\lambda t)^k}{k!}e^{-\lambda t}\right]+1$$

Problem 2.3

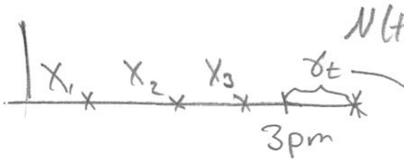
Limousines depart from the railway station to the airport from early morning to late at night.

$X_1, X_2, \dots$ interdeparture times i.i.d $X_i \sim U[10,20]$ min

Arrive at railway station at 3pm.

Est. mean & S.D. of waiting time for limousine.

$N(t)$ – the renewal process generated by $X_1, X_2, \dots$



2.1.8 $E[\gamma_t] \approx \frac{\mu_2}{2\mu_1}$ $E[\gamma_t^2] \approx \frac{\mu_3}{3\mu_1}$ $\mu_i = E[X_i^i]$

$E[\gamma_t] \approx 7.78$ $E[\gamma_t^2] \approx 83.34 \Rightarrow Var(\gamma_t) = E[\gamma_t^2] - E[\gamma_t]^2$

$< \mu_1 = 15$ $= 22.81$

$SD(\gamma_t) = \sqrt{Var(\gamma_t)} = 4.78$

Can $E[\gamma_t]$ be larger than the average interoccurence time?

Ex 2.1.4 [The waiting time paradox]

$$\frac{\mu_2}{2\mu_1} = \frac{\mu_1}{2}\left[\frac{\mu_2}{\mu_1^2} - 1 + 1\right] = \frac{\mu_1}{2}\left[\frac{\mu_2 - \mu_1^2}{\mu_1^2} + 1\right] = \frac{\mu_1}{2}\left[\underbrace{\frac{Var(X_n)}{\mu_1^2}}_{C_X^2 \text{ – variation coeff}} + \right]$$

$C_X^2 > 1$: $\frac{\mu_2}{2\mu_1} > \mu_1$ $C_X^2 < 1$: $\frac{\mu_2}{2\mu_1} < \mu_1$

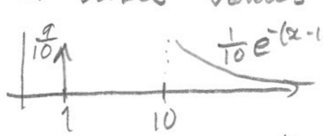
In the long run $E[\gamma_t] > \mu_1$ if $C_X^2 > 1$

$E[\gamma_t] < \mu_1$ if $C_X^2 < 1$

Ex irregularities in distr.

$X_1, X_2, \dots$ inter occurence times, iid distr. like X.

X takes values in $[10, \infty)$ or equals 1.



$$f_X(x) = \frac{1}{10}e^{-(x-10)}\theta(x-10) + \delta(x-1)\cdot\frac{9}{10}$$

$$\mu_1 = 1\cdot\frac{9}{10} + \int_{10}^{\infty} x\cdot\frac{1}{10}e^{-(x-10)}dx = \frac{9}{10} + \frac{1}{10}\cdot 11 = 2.$$

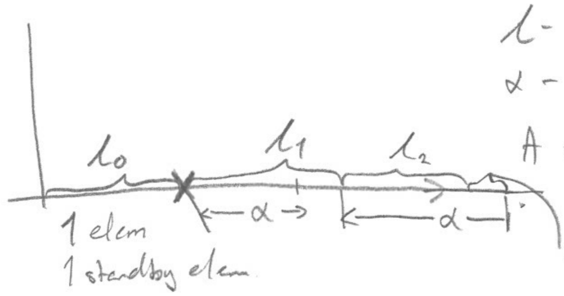
large t, $E[\gamma_t] \approx 3,27$ $C_X^2 = 2.27 > 1$

Example 3

l - lifetime

α - repair time of element

A repaired element works like a new one.



$$N = \{\min n \geq 1 : l_1 \geq \alpha, \dots, l_{n-1} \geq \alpha, l_n < \alpha\}$$

$$P\{N=1\} = P\{L_1 < \alpha\} = F(\alpha) \quad , \quad F(x) \text{ distribution funct. of } l_0$$

$$P\{N=n\} = P\{L_1 \geq \alpha, \dots, L_{n-1} \geq \alpha, L_n < \alpha\} = \{\text{i.i.d}\}$$

$$= P\{L_1 \geq \alpha\} \cdot P\{L_2 \geq \alpha\} \cdot \ldots \cdot P\{L_{n-1} \geq \alpha\} \cdot P\{L_n < \alpha\}$$

$$= [1 - F(\alpha)]^{n-1} \cdot F(\alpha) \quad n \in \mathbb{N}.$$

Geom. r.v $\quad N \sim Geom(F(\alpha))$

C - lifetime of syst.

$$C = l_0 + l_1 + l_2 + \ldots + l_N = l_0 + \sum_{i=1}^{N} l_i =$$

$$\{N=n\} = \{l_1 \geq \alpha, \dots, l_{n-1} \geq \alpha, l_n < \alpha\}$$

indep of l_{n+1} Walds equality

$$\mathbb{E}\left[\sum_{i=1}^{N} l_i\right] = E[N] \cdot E[l_0]$$

or any l_i : μ

$$E[C] = \mu\Big[1 + \underbrace{\frac{1}{F(\alpha)}}_{E[N]}\Big]$$

12/11 Stokastiske processer I

2.2 Renewal reward processes

Ex: Consider the (S,s) inventory system

I_n - the inventory level at the beginning of week n.

$I_0 = S$, $\{I_n, n \geq 0\}$

The process regenerates itself at the beginning of a replenishing week.

Ex: Consider a single server system.

- Customers arrive according to a renewal process.
- At $t=0$ there is a customer and the server is idle.
- The system regenerates itself any time a customer finds the server idle.

Def. 2.2.1

Let T be $[0,\infty)$ or $\{0,1,2,\dots\}$

A random process $\{X(t), t \in T\}$ is said to be regenerative if there exists a random epoch S_1 s.t.

a) ~~$X(t)$~~ $\{X(t+S_1), t \in T\}$ is indep. of $\{X(t), t \in [0, S_1]\}$

b) $\{X(t+S_1), t \in T\}$ has the same prob. distr. as $\{X(t), t \in T\}$

It follows from the def. that there exists further epochs of regeneration $S_2, S_3, \dots$ $C_i = S_i - S_{i-1}$ a cycle

- $C_1, C_2, C_3, \dots$ are iid r.v.

The renewal process associated w/ $\{X(t), t \geq 0\}$, assuming $E[C_1] < \infty$

let $N(t)$ be the renewal process generated by $C_1, C_2, \dots$

- $N(t)$ = the # of regenerations up to the moment t.

Assume in the initial example there is a cost of h for any inspection of the inventory level. The cost is accumulated in time

At the end of the cycle this lumps into a reward.

R_n - the reward in one cycle.

- $\{R(t), t \geq 0\}$ ~~$R(t)$~~ the renewal-reward process associated w/ $\{X(t), t \geq 0\}$

  total reward at time t.

Thm 2.2.1 [The renewal-reward thm]

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R_1]}{E[C_1]} \quad \text{w/ probability 1}$$

the long run average rewards/unit of time.

Comments

- The result is v. important in practice.
  - The reward can be predicted in the long run
  - $R(t) \approx t \cdot \frac{E[R_1]}{E[C_1]}$ (linear)
  - $R(t) \approx E[R_1] \underbrace{\frac{t}{E[C_1]}}$ average # cycles in $[0,t]$ [lemma 2.2.2]

The expected value version

$$\lim_{t\to\infty} E\left[\frac{R(t)}{t}\right] = \frac{E[R_1]}{E[C_1]}$$

TQ lemma 2.2.2

$\{X(t), t\geq 0\}$ is a regenerative process w/ cycles $C_1, C_2, \dots$

$E[C] < \infty$.

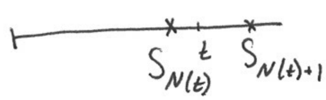
- $N(t)$ - the renewal process assoc. w/ $\{X(t), t\geq 0\}$

Then $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E[C_1]}$ w/ prob. 1.

Proof:

Note: $\lim_{t\to\infty} N(t) = \infty$

$$C_1 + C_2 + \dots + C_{N(t)} \leq t < C_1 + C_2 + \dots + C_{N(t)+1}$$



divide by $N(t)$

$$\frac{C_1 + C_2 + \dots + C_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{C_1 + C_2 + \dots + C_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)}$$

when $t \to \infty$, $N(t) \to \infty$ and by the SLLN

$$\lim_{t\to\infty} \frac{C_1 + \dots + C_{N(t)}}{N(t)} = E[C_1] \text{ w/ prob. 1.}$$

$$\lim_{t\to\infty} \frac{C_1 + \dots + C_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} = E[C_1] \text{ w/ prob. 1.}$$

$$\Rightarrow \lim_{t\to\infty} \frac{t}{N(t)} = E[C_1] \quad \left(1 + \frac{1}{N(t)}\right) \to 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E[C_1]} \quad \square$$

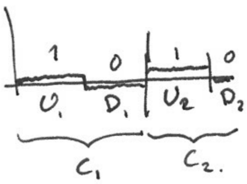
Ex: 2.2.1

- A machine can be up or down. $U_1, U_2, \dots$ lengths of "up"-periods
  $D_1, D_2, \dots$ lengths of "down"-periods.

$\{U_n\}, \{D_n\}$ each i.i.d, but U_n, D_n not ass. independent

At $t=0$, state is up.

What is the long run fraction of time the machine is down



$$X(t) = \begin{cases} 1 & \text{if "up"} \\ 0 & \text{if "down"} \end{cases}$$

regenerates, $C_1 = U_1 + D_1$

$\vdots$

$C_i = U_i + D_i$

Assume a cost of rate 1, while machine is down.

R_1 - the cost in 1 cycle, $E[R_1] = E[D_1]$

- $E[C_1] = E[U_1] + E[D_1]$
- $R(t)$ accumulated cost up to time t.

By renewal-reward thm.

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R_1]}{E[C_1]} = \frac{E[D_1]}{E[U_1]+E[D_1]}$$

the long run average cost / time unit on the long run fraction of time the machine is down.

- Let $\mathcal{B}$ be a set of states for $\{X(t), t \geq 0\}$
  define a process $I_B(t) = \begin{cases} 1 & \text{if } X(t) \in B \\ 0 & \text{if } X(t) \notin B \end{cases}$

- T_B be the amount of time the process spends in B during a cycle

$$T_B = \int_0^{S_1} I_B(t)\,dt, \quad T \in [0,\infty)$$

$$\sum_{k=0}^{S_1} I_B(k) \quad T = \{0,1,\dots\}$$

In given setting:

Thm 2.2.3

W prob 1: $\lim_{t\to\infty} \frac{1}{t} \int_0^t I_B(u)\,du = \frac{E[T_B]}{E[C_1]} \quad T = [0,\infty)$

$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} I_B(k) \quad T = \{0,1,2,\dots\}$

The expected value version

$$0 \le \frac{1}{t}\int_0^t I_B(u)\,du \le 1$$

$$0 \le \frac{1}{n}\sum_{k=0}^{n} I_B(k) \le 1$$

by the bounded convergence thm

$$\lim_{t\to\infty} E\left[\frac{1}{t}\int_0^t I_B(u)\,du\right] = \frac{E[T_B]}{E[C_1]}$$

$$\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{k=0}^{n} I_B(k)\right] = \frac{E[T_B]}{E[C_1]}$$

compute the expectation:

① $$E\left[\frac{1}{t}\int_0^t I_B(u)\,du\right] = \frac{1}{t}\int_0^t E[I_B(u)]\,du = \frac{1}{t}\int_0^t P\{X(u)\in B\}\,du$$

$$E\left[\frac{1}{n}\sum_{k=0}^{n} I_B(k)\right] = \frac{1}{n}\sum_{k=0}^{n} P\{X(k)\in B\}$$

assume for the moment that

$$\exists \lim_{t\to\infty} P\{X(t)\in B\} \wedge \exists \lim_{n\to\infty} P\{X(n)\in B\}$$

Holds when
C_1 has cont. parts if $T\in[0,\infty)$
The values of C_1 are relatively prime, $T=\{0,1,2,\dots\}$

$$\left[\lim_{t\to\infty}\frac{1}{t}\int_0^t g(u)\,du = c\right]$$

① $$\lim_{t\to\infty}\frac{1}{t}\int_0^t P\{X(u)\in B\}\,du = \lim_{t\to\infty} P\{X(t)\in B\}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} P\{X(k)\in B\} = \lim_{n\to\infty} P\{X(n)\in B\}$$

13/11 Stokastiska processer

- $\{X(t), t \in T\}$ a reg. process $C_1, C_2, \ldots$ cycles $E[C] < \infty$

Assumptions when $T = [0, \infty)$ C_1 has a contin. part.

$T = \{0, 1, 2, \ldots\}$ the values of C_1 are relatively prime

- B a set of possible states.

T_B - time of one cycle spent in B.

<u>Thm 2.2.4</u>

$$\lim_{t \to \infty} P\{X(t) \in B\} = \frac{E[T_B]}{E[C_1]} = \{T \text{ continuous}\}\left[= \lim_{t \to \infty} \frac{1}{t} \int_0^t I_B(u)\,du \text{ w/ prob. } 1\right]$$

$$\lim_{n \to \infty} P\{X(n) \in B\} = \frac{E[T_B]}{E[C_1]} = [T \text{ discrete}]\left[= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} I_B(k) \text{ w/ prob. } 1\right]$$

Poisson arrivals see time averages [PASTA]

- In a system analysis we sometimes need to know
  - a| the long run fraction of time the system is in a ~~syste~~ given state
  - b| the long run fraction of time an arrival finds the system in a given state

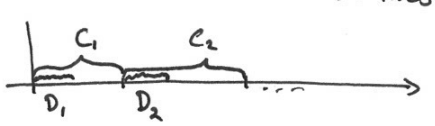
$\Rightarrow$ The answers aren't necessarily the same.

<u>Ex</u>

A server system (1). Customers arrive w/ i.i.d interarrival times $C_1, C_2, \ldots$ $C_1 \sim U[1,2]$ min

Service times are i.i.d r.v $D_1, D_2, \ldots$ $D_1 \sim U\left[\frac{1}{4}, \frac{3}{4}\right]$ min.

At $t=$ a customer arrives and the system is idle.



$$X(t) = \begin{cases} 1 & \text{if system is busy} \\ 0 & \text{if system idle} \end{cases}$$

a regenerating process.

$C_1 \sim U[1,2]$, $E[C_1] = \frac{3}{2}$

The time $X(t) = 1$ is 1 cycle D_1. The long fraction of time the system is busy $= \frac{E[D_1]}{E[C_1]} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$ At the same time any customer finds the system idle.

Ex 2.4.1

Jobs arrive at a workstation according to a P.P of rate λ

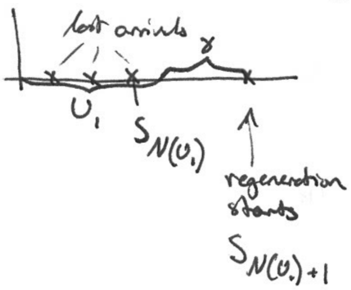
An arriving job is accepted only if the workstation is idle. (otherwise lost)

Processing time of jobs are i.i.d r.v. w/ mean value β.

a) What is the long run fraction of time the system is busy?

b) What is the long run fraction lost jobs (arrivals finding the system busy)?

a) $U_1, U_2, \ldots$ Processing times (i.i.d) $E[U_i] = \beta$



$I(t) = \begin{cases} 1 & \text{if syst. busy} \\ 0 & \text{—"— idle} \end{cases}$

a regen. proc.

$C_1 = U_1 + \gamma$. $E[C_1] = E[U_1 + \gamma] = \beta + E[\gamma] = \beta + \frac{1}{\lambda}$

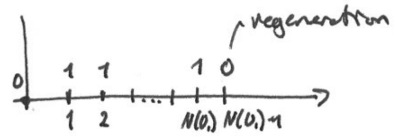
$E[\gamma | U_1 = t] = \frac{1}{\lambda}$, $E[\gamma] = E[E[\gamma | U_1]] = \frac{1}{\lambda}$

The long run fraction of time the system is busy

$= \dfrac{E[U_1]}{E[C_1]} = \dfrac{\lambda\beta}{\lambda\beta + 1}$

b) define: $n = 0, 1, \ldots$ $I_n = \begin{cases} 1 & \text{if syst. is busy prior to the n-th arrival} \\ 0 & \text{otherwise} \end{cases}$

$I_0 = 0$



I_n - reg. process w/

$\tilde{C}_n = N(U_1) + 1$ $E[\tilde{C}_n] \stackrel{+1}{=} E[N(U_1)] \stackrel{+1}{=} \lambda \cdot \beta \cdot + 1$

$E[N(U_1) | U_1 = t] = \lambda t$, $E[N(U_1) | U_1] = \lambda U_1$

$E[N(U_1)] = E[\lambda U_1] = \lambda \cdot \beta$

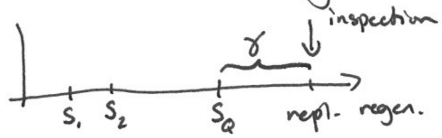
$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} I_k = \{\text{Thm. 2.2.3}\} = \dfrac{\lambda\beta}{\lambda\beta + 1}$

Ex: 2.4.2

- Cust. arrive as a P.P. w/ rate λ. Each cust. requests 1 unit of the product.
- Each unsatisfied request is lost.
Inspection is done according to a P.P w/ rate μ (indep of cust. arrival)
- Replenishment takes place only when no stock remains. and the inv. is then raised to the initial level Q.

a) What is the long run fraction of time the system is out of stock?

b) What is the long run fraction of requests which are lost.



$I(t) = \begin{cases} 1 & \text{if syst is out of stock at } t. \\ 0 & \text{otherwise} \end{cases}$

cycle $C_1 = S_Q + \gamma$ $E[C_1] = \frac{Q}{\lambda} + E[\gamma]$

$E[\gamma | S_Q = t] = \frac{1}{\mu}$: $E[\gamma] = \frac{1}{\mu}$

In one cycle the system is out of stock time γ.
The long run fraction of time the system is out of stock $= \frac{E[\gamma]}{E[C_1]}$

$$= \frac{\frac{1}{\mu}}{\frac{Q}{\lambda}+\frac{1}{\mu}}$$

b) $N_1 =$ the # of lost customers in 1 cycle.

$$E[N_1 \mid \gamma = t] = \lambda t, \quad E[N_1 \mid \gamma] = \lambda\gamma \quad E[N_1] = \lambda E[\gamma] = \frac{\lambda}{\mu}$$

The # cust. in 1 cycle is $Q + N_1$

The long run fraction of cust which find the system out of stock (lost cust) $= \frac{\lambda/\mu}{Q + \frac{\lambda}{\mu}}$

PASTA in a broader context

$\{N(t), t \geq 0\}$ a PP of rate λ describes arrivals to a system.

$\{X(t), t \geq 0\}$ a cont. time process describes the system.

Lack of anticipation assumption

$\{N(t), t > u\}$ and $\{X(t), t \leq u\}$ are indep. for any $u > 0$

~~Introduce~~

- B - set of states of $X(t)$, define $I_B(t) = \begin{cases} 1 & \text{if } X(t) \in B \\ 0 & \text{otherwise} \end{cases}$

Introduce X_n - the state of system just prior to the n-th arrival.

$$I_B(n) = \begin{cases} 1 & X_n \in B \\ 0 & \text{otherwise} \end{cases}$$

Thm 2.4.1 [PASTA]

a) $E\left[\begin{matrix}\text{the \# of arrivals in } (0,t) \\ \text{finding the system in state of } B\end{matrix}\right] = \lambda E\left[\int_0^t I_B(u)\,du\right]$

b) $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} I_k(B) = \lim_{t \to \infty} \frac{1}{t} \int_0^t I_B(u)\,du$

Thm 2.4.1 can be specified for a regenerative process $\{X(t), t \geq 0\}$

Let $\{X(t), t \geq 0\}$ be a reg. process w/ $E[C_1] < \infty$

B - set of states for process. T_B - time spent in B for a cycle

N_B - # of arrivals in 1 cycle who find the system in B.

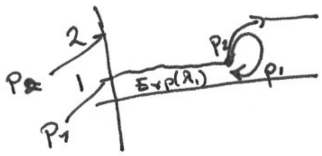
Corollary

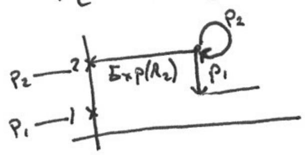
$E[N_B] = \lambda E[T_B]$ (prove on your own).

17/11 Stokastiske processer.

Pr. 2.6 $\{X(t),\ t \geq 0\}$ has 2 states 1, 2.

$P\{X(0)=i\} = p_i \quad i = 1,2.$





a) Find the long. running fraction of time the process is in state i.

b) does $\lim_{t\to\infty} P\{X(t)=i\}$ exist?

Consider a renewal process gen. by interoccurence times w/ density function $\lambda p_1 e^{-\lambda_1 t} + p_2 \lambda e^{-\lambda_2 t}$. Show that $\lim_{t\to\infty} P\{\gamma_t > x\} = \dfrac{p_1\lambda_2 e^{-\lambda_1 x} + p_2\lambda_1 e^{-\lambda_2 x}}{p_1\lambda_2 + p_2\lambda_1}$

Solution:

a) $X(t)$ is a regen. process C_1 - the length of first cycle.

$$P\{C_1 \leq x\} = P\{C_1 \leq x \mid X(0)=1\} p_1 + P\{C_1 \leq x \mid X(0)=2\} p_2$$

$$= \left(1-e^{-\lambda_1 x}\right)p_1 + \left(1-e^{-\lambda_2 x}\right)p_2$$

$$f_{C_1}(x) = p_1\lambda_1 e^{-\lambda_1 x} + p_2\lambda_2 e^{-\lambda_2 x}$$

$$E[C_1] = \int_0^\infty x f_{C_1}(x)\,dx = p_1 E\left[\lambda_1 e^{-\lambda_1 x}\right] + p_2 E\left[\lambda_2 e^{-\lambda_2 x}\right] = \frac{p_1}{\lambda_1} + \frac{p_2}{\lambda_2}$$

Assume th. system earns a reward (1) for 1 time unit spent in state 1, 0 otherwise.

R_1 = the reward in first cycle in state one.

$$E[R_1] = E[R_1 \mid X(0)=1]p_1 + \underbrace{E[R_1 \mid X(0)=2]}_{=0} p_2 = \frac{p_1}{\lambda_1}$$

By t. renewal-reward thm:

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R_1]}{E[C_1]} = \frac{p_1/\lambda_1}{p_1/\lambda_1 + p_2/\lambda_2}$$ = long run fraction of time $X(t)$ is in state 1.

Since C_i is continuous, $\lim_{t\to\infty} P\{X(t)=1\} = \lim_{t\to\infty} \frac{R(t)}{t} = \dfrac{p_1}{p_1 + p_2\frac{\lambda_1}{\lambda_2}}$

b) $N(t)$ - a renewal process associated w/ $\{X(t),\ t\geq 0\}$

$C_1 \quad C_2 \quad \gamma_t \quad t$

$$P\{\gamma_t > x\} = \cancel{P\{\gamma_t > x \mid X(0)=1\}p_1 + P\{\gamma_t > x \mid X(0)=2\}p_2}$$

$$\cancel{= e^{-\lambda_1 x} p_1}$$

$$= P\{\gamma_t > x \mid X(t)=1\} P\{X(t)=1\} + P\{\gamma_t > x \mid X(t)=2\} P\{X(t)=2\}$$

$$= e^{-\lambda_1 x} P\{X(t)=1\} + e^{-\lambda_2 x} P\{X(t)=2\} \longrightarrow \frac{\lambda_2 p_1 e^{-\lambda_1 x} + p_2\lambda_1 e^{-\lambda_2 x}}{p_1\lambda_2 + p_2\lambda_1}$$

$$\xrightarrow{t\to\infty} \frac{\frac{p_1}{\lambda_1}}{\frac{p_1}{\lambda_1}+\frac{p_2}{\lambda_2}} \qquad \frac{p_2/\lambda_2}{\frac{p_1}{\lambda_1}+\frac{p_2}{\lambda_2}}$$

Pr. 2.12

Waste is temporarily stored in a factory

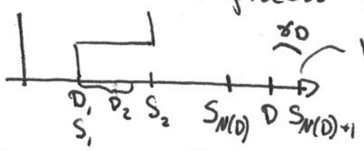
- $D_1, D_2, \dots$ – the weekly production of waste i.i.d. r.v.

Controls occur at the end of each week, if the total amount of waste is $> D$ it is removed, otherwise its kept for next week.

Cost k for each time waste is removed, $v > 0$ for each unit of waste in excess of D.

- Find long run average cost/time unit.

– $N(t)$ r. renewal process gen. by $D_1, D_2, \dots$



$\{X_n, n = 0, 1, \dots\}$ – the regen. process w/ $C_1 = N(D) + 1$

$E[C_1] = E[N(D)] + 1 =$

γ_D – the excess amount in week $N(D)+1$

$$\gamma_D = S_{N(D)+1} - D = \sum_1^{N(D)+1} D_i - D$$

$$E\left[\sum_1^{N(D)+1} D_i\right] =$$

$$\{N(D)+1 = n\} = \{N(D) = n-1\} = \{S_{n-1} \le D, S_n > D\}$$ $\Rightarrow$ Walds eqn OK!

indep of D_{n+1}, D_{n+2}

$$E\left[\sum_1^{N(D)+1} D_i\right] = E[N(D)+1]\underbrace{E[D_1]}_{\mu_1} = (M(D)+1)\mu_1$$

$$E[\gamma_D] = [M(D)+1]\mu_1 - D$$

$$\frac{v((M(D)+1)\mu_1 - D) + k}{(M(D)+1)\mu_1} = \text{the long run cost/time unit}$$

Further use r. app. for $M(D)$.

2.13 a variant

- Messages arrive according to a renewal process w/ mean interarrival time $\frac{1}{\lambda}$
- Messages are stored in a buffer, which is emptied when M messages have arrived.

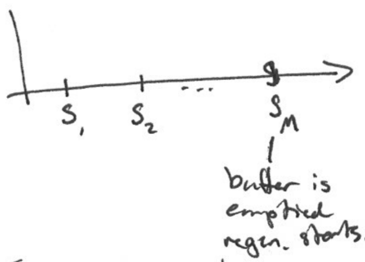
Costs of:

$k>0$ for emptying the buffer

$v>0$ for each unit of time a message is in the buffer

Find M which minimises the long-run average cost/time unit.

$S_1, S_2, \ldots$ arrival points of messages



$X(t) = \#$ of messages in buffer at time t.

regenerated

$C_1 = S_M$

$E[C_1] = \frac{M}{\lambda}$

Total waiting time in the first cycle. $\sum_{k=1}^{M-1}(S_M - S_k)$

$$E\left[\sum_{k=1}^{M-1}(S_M - S_k)\right] = \sum_{k=1}^{M-1}\left(E[S_M] - E[S_k]\right) = \frac{1}{\lambda}\sum_{k=1}^{M-1}(M-k) = \frac{1}{\lambda}\sum_{j=1}^{M-1} j = \frac{1}{\lambda}\cdot\frac{(M-1)M}{2}$$

The average cost in one cycle.

$$\frac{v(M-1)M}{2\lambda} + k$$

The long run average cost/time unit: $= \dfrac{\frac{v(M-1)M}{2\lambda} + k}{M/\lambda}$

$$= \frac{v(M-1)}{2} + \frac{\lambda k}{M} = g(M) \quad M \geq 1$$

$g'(x) = \frac{v}{2} - \frac{\lambda k}{x^2}$ $\quad g'(x) = 0 \iff \frac{2}{v} = \frac{x^2}{\lambda k} \Rightarrow x_0 = \pm\sqrt{\frac{2\lambda k}{v}}$

$g''\left(\frac{2\lambda k}{v}\right) > 0$: $M = \left\lfloor\sqrt{\frac{2\lambda k}{v}}\right\rfloor$ or $\left\lceil\sqrt{\frac{2\lambda k}{v}}\right\rceil$ lower or upper integer part of x_0.

<u>More on the memoryless property of Exp</u>

$X \sim Exp(\lambda)$ we know: $P\{X > t+h \mid X > t\} = P\{X > h\}$ $\forall t, h > 0$

What if t is random? Y ind of X. $Y \sim f_Y(y)$ $Y \geq 0$.

$$P\{X > Y+h \mid X > Y\} = \frac{P\{X > Y+h\}}{P\{X > Y\}} =$$

$$\left\{P\{X > Y+h\} = \int_{y=0}^{\infty} P\{X > Y+h \mid Y=y\} f_Y(y)\,dy = \int_0^{\infty} e^{-\lambda(y+h)} f_Y(y)\,dy = e^{-\lambda h}\underbrace{\int e^{-\lambda y} f_Y(y)\,dy}_{P\{X>Y\}}\right\}$$

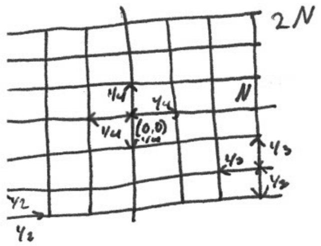
$$= \frac{e^{-\lambda h}\,P\{X > Y\}}{P\{X > Y\}} = e^{-\lambda h}$$

Read Ex1.1.5

19/11 Stochastic processes (Markov) Ch3

Discrete time Markov chains

Ex 3.1.4
random walk in a grid in $2N \times 2N$ quadrant.



Def: 3.1.1.
a) The s.p. $\{X_n, n=0,1,2,\dots\}$ w/ state space I is a Markov chain if
for $n=0,1,2,\dots$ $P\{X_{n+1}=i_{n+1} \mid X_n=i_n, X_{n-1}=i_{n-1},\dots,X_0=i_0\} = P\{X_{n+1}=i_{n+1} \mid X_n=i_n\}$
for any states $i_0, i_1, \dots, i_{n+1} \in I$

b) ~~Ex~~ ~~P{X_{n+1}~~ If the one-step transition doesn't depend on n the Markov chain is homogeneous.
$P\{X_{n+1}=i_{n+1} \mid X_n=i_n\}$

Notations
$I=\{1,2,\dots\}$
$P_{ij} = P\{X_{n+1}=j \mid X_n=i\},\ i,j \in I$ $\qquad P=[P_{ij}]_{i,j\in I}$ is called the onestep transitionmatrix

$$\sum_{j\in I} P_{ij} = 1,\ \forall i \in I$$

The MCh $\{X_n, n=0,1,\dots\}$ w/ state I is completely determined by
- the distr. of X_0
- P

$$P\{X_1=i\} = \sum_{j\in I} \underbrace{P\{X_1=i \mid X_0=j\}}_{P_{ji}} \underbrace{P\{X_0=j\}}_{\text{known}}$$
⋮
etc.

Ex 3.1.2
A stock control problem.
- Initially a stock of S pliers
- Weekly demands are independent, Poi(λ)-distr. r.v.

Each demand which occurs when the shop is out of stock is lost.
The owner inspects the stock in the beginning of the week, using (S,s) control rule.

X_n = the # of pliers in stock in the beg. of week n.
$\{X_n, n\geq 0\}$; $I=\{0,1,\dots S\}$ $\quad X_{n+1}$ is (probabilistically) determined X_n and the demand in week n.

$\{X_n, n=0,1,\dots\}$ is a MCh!

$P_{ij} = P\{X_{n+1}=j \mid X_n = i\} \quad i,j \in I$

<u>i<s</u>

The possible states at the start of the next week $j=0,1,2,\dots$

$$P_{i0} = P\{X_{n+1}=0 \mid X_n=i\} = \sum_{k=s}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda}$$

$j=1,2,\dots$

$$P_{ij} = P\{X_{n+1}=j \mid X_n=i\} = \frac{\lambda^{s-j}}{(s-j)!} e^{-\lambda}$$

$i \geq s$ The possible states at the start of the next week

$j=0,1,\dots,i$

$j=0$ $$P_{i0}=P\{X_{n+1}=0 \mid X_n=i\} = P\{\text{demand in week } n \geq i\} = 1 - \sum_{k=0}^{i-1} \frac{\lambda^k}{k!} e^{-\lambda}$$

$j=1,2,\dots$

$$P\{X_{n+1}=j \mid X_n=i\} = \frac{\lambda^{j-i}}{(j-i)!} e^{-\lambda}$$

3.2 Transient analysis

$\{X_n, n=0,1,\dots\}$ w/ I an MCh. $P=[P_{ij}]_{i,j\in I}$

$$P_{ij}^{(n)} = P\{X_n=j \mid X_0=i\}$$

$\{P_{ij}^{(n)}\}_{i,j\in I}$ – the n-th step transient probability.

$P_{ij}^{(1)} = P_{ij}$

For convenience $P_{ij}^{(0)} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ (Kronecker delta)

T.Q. Thm 3.2.1 Chapman – Kolmogoroff equation

For $n,m=0,1,2,\dots$

$$P_{ij}^{(n+m)} = \sum_{k\in I} P_{ik}^{(n)} P_{kj}^{(m)} \qquad (3.2.1)$$

<u>Proof:</u>

- When $m=0$ or $n=0$ 3.2.1 true
- $m,n \geq 1$

$$\{X_{\underset{\geq n+1}{n+m}}=j\} = \underbrace{\bigcup_{k\in I}\{X_{n+m}=j, X_n=k\}}_{\text{mutually exclusive}}$$

$$P\{X_{n+m}=j \mid X_0=i\} = \sum_{k\in I} P\{X_{n+m}=j, X_n=k \mid X_0=i\} = \sum_{k\in I} \frac{P\{X_{n+m}=j, X_n=k, X_0=i\}}{P\{X_0=i\}}$$

$$= \sum_{k\in I} P\{X_{n+m}=j \mid X_n=k, \underbrace{X_0=i}_{\text{negl. bec. proc. MCh.}}\} \cdot \underbrace{\frac{P\{X_n=k, X_0=i\}}{P\{X_0=i\}}}_{P\{X_n=k \mid X_0=i\}} = \sum_{k\in I} P_{kj}^{(m)} \cdot P_{ik}^{(n)}$$

$\square$

In particular [m=1]

$$P_{ij}^{(n+1)} = \sum_{k\in I} P_{ik}^{(n)} P_{kj} \quad i,j \in I$$

n=1

$$P_{ij}^{(2)} = \sum_{k\in I} P_{ik} P_{kj} \qquad P = \begin{bmatrix} P_{11} & P_{12} & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

elem of col j.

elem. of row i

$$P^{(2)} = [P_{ij}^{(2)}] ; \quad P^{(2)} = P^2$$

n=2

$$P_{ij}^{(3)} = \sum_{k\in I} P_{ik}^{(2)} P_{kj} \qquad P^{(3)} = P^{(2)} \cdot P = P^3 \Rightarrow P^{(n)} = P^n$$

Ex: 3.2.1

Weather as a MCh.

- On the Island of Hope each day is {sunny, cloudy, rainy}

$$I = \{1, 2, 3\}$$
S C R

- Next day weather depends (only) on the weather in the present day.

The one-step trans. prob. matrix

$$P = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.5 & 0.25 & 0.25 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

How often is the weather S, C, R over a long period of time?

A different question is answered: Is there a limit $\lim_{n\to\infty} P\{X_n = i\}$ $i \in I$

$$P\{X_{n+3} = 1 | X_n = 2\} = P_{21}^{(3)} = 0.59125$$

$$P^3 = \begin{bmatrix} 0.6015 & 0.16825 & 0.23025 \\ 0.59125 & 0.175625 & 0.233125 \\ 0.5855 & 0.17955 & 0.234756 \end{bmatrix}$$

$$P^{12} \approx P^{13} \approx P^{14}$$

$$P^{12} = \begin{bmatrix} 0.596 & 0.172 & 0.232 \\ 0.596 & 0.172 & 0.232 \\ 0.596 & 0.172 & 0.232 \end{bmatrix}$$

P{sunny day in 12 days} = 0.596 · P{sunny today} + 0.596 P{Cloudy today} + 0.596 P{rainy day today} = 0.596

20/11 Stochastic processes

3.2.1 Absorbing states. (time homogeneous, discrete time MCh)

$\{X_n, n=0,1,\dots\}$; I; $P=[P_{ij}]_{i,j\in I}$

Def
$r\in I$ is absorbing if $P_{rr}=1$

r - absorbing, $X_0=0$ $0\in I$
U - the absorption time
$P\{U=m\}=?$ $m=1,2,\dots$

Consider $1-P_{or}^{(m)}=1-P\{X_m=r|X_0=0\}=P\{X_m\neq r|X_0=0\}$
$=P\{X_1\neq r,\dots,X_m\neq r|X_0=0\}=P\{U>m\}$

$$P\{U=m\}=P\{U>m-1\}-P\{U>m\}=P_{or}^{(m)}-P_{or}^{(m-1)}$$

3.2.2 Mean first passage times

$\{X_n, n=0,1,2,\dots\}$, I, P

Assumptions
- I is a finite set.
- $\exists r\in I: \forall i\in I\ \exists n_i=n(i),: P_{ir}^{(n_i)}>0$ [r is accessible from any $i\in I$]

Define $\tau=\min\{n\geq 1; X_n=r\}$ the time for first visit to r. [whatever the starting state]

Introduce for $i\in I$

$\mu_{ir}=E[\tau|X_0=i]$
mean first visit time from i to r.

$\mu_{rr}=E[\tau|X_0=r]$
mean first visit time from r to r.

T.Q: Lemma A

In the above settings: $\mu_{rr}=1+\sum_{\substack{j\in I\\ j\neq r}} P_{rj}\mu_{jr}$ [3.2.3]

Proof:

$$\mu_{rr}=E[\tau|X_0=r]=\sum_{k=1}^{\infty}k\,\underbrace{P\{\tau=k|X_0=r\}}$$

$k=1$: $P\{\tau=1|X_0=r\}=P\{X_1=r|X_0=r\}=P_{rr}$

$k\geq 2$ $\{\tau=k\}=\bigcup_{\substack{j\in I\\ j\neq r}}\{\tau=k,X_1=j\}$ disjoint ev., $P\{\tau=k|X_0=r\}=\sum_{\substack{j\in I\\ j\neq r}}P\{\tau=k,X_1=j|X_0=r\}$ By MCh.

$$=\sum_{\substack{j\in I\\ j\neq r}}\frac{P\{\tau=k,X_1=j,X_0=r\}}{P\{X_0=r\}}=\sum_{\substack{j\in I\\ j\neq r}}P\{\tau=k|X_1=j,X_0=r\}\frac{P(X_1=}{P\{}$$

$$= \sum_{\substack{j \in I \\ j \neq r}} P\{\tau = k \mid X_1 = j\} \, P\{X_1 = j \mid X_0 = r\}$$

27/11 Stokastiske processes.

3.2.3 Transient and recurrent states

$\{X_n, n \geq 0\}$ a MCh. I, P Define for $n = 1, 2, \ldots$ $i,j \in I$

$$f_{ij}^{(n)} = P\{X_n = j, X_k \neq j, n-1 \geq k \geq 1 \mid X_0 = i\}$$

the first passage time probability.

$$S_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} \qquad 0 \leq f_{ij} \leq 1$$

the probability that the process visits j given that it starts in i.

Def: 3.2.2 $i \in I$ is transient if $f_{ii} < 1$

$i \in I$ is recurrent if $f_{ii} = 1$

Equivilent def:

$Q_{ii} = P\{X_n = i$ for infinitely many $n \mid X_0 = i\}$

$i \in I$ transient iff $Q_{ii} = 0$

$i \in I$ recurrent iff $Q_{ii} = 1$

Lemma B (3.2.10)

$\{X_n, n \geq 0\}$ a MCh $i \in I$, i is transient $\iff \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$

Sketch of proof:

$$I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{otherwise} \end{cases}$$

$N = \sum_{n=1}^{\infty} I_n$ = the # of visits to i after $n = 0$

$E[N \mid X_0 = i]$ can be computed in 2 ways.

(1) $E\left[\sum_{n=1}^{\infty} I_n \mid X_0 = i\right] = \sum_{n=1}^{\infty} E[I_n \mid X_0 = i] = \sum_{n=1}^{\infty} 1 \cdot P\{I_n = 1 \mid X_0 = i\} = \sum_{n=1}^{\infty} P_{ii}^{(n)}$

(2) $E[N \mid X_0 = i] = \sum_{k=1}^{\infty} P\{N \geq k \mid X_0 = i\} = \sum_{k=1}^{\infty} \Big[1 - \sum_{n=0}^{k-1} \underbrace{P\{N = n \mid X_0 = i\}}_{f_{ii}^{n}(1 - f_{ii})}\Big]$

$N \in \mathbb{N}$

$$= \sum_{k=1}^{\infty} \left[1 - (1 - f_{ii}^k)\right] = \sum_{k=1}^{\infty} f_{ii}^k$$

$$\Rightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} = \sum_{n=1}^{\infty} f_{ii}^n$$

i is transient $\iff f_{ii} < 1 \iff \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$

Corollary:

If j is transient then $\lim_{n\to\infty} P_{ii}^{(n)} = 0$

More is true.

Lemma 3.2.3

If i is transient then $\lim_{n\to\infty} P_{ji}^{(n)} = 0$ for any $j \in I$

In the long run a transient state "disappears"

What about the recurrent state?

Prelim:

Suppose j is recurrent $(\tau = \min\{n \geq 1; X_n = j\})$ first visit time to j.

$$\mu_{jj} = E[\tau | X_0 = j] = \sum_{n=1}^{\infty} n \cdot P\{\tau = n | X_0 = j\} = \sum_{n=1}^{\infty} n f_{jj}^{(n)}$$

$$\underbrace{P\{X_n = j, X_k \neq j, n-1 \geq k \geq 1 | X_0 = j\}}_{f_{jj}^{(n)}}$$

2 types of recurrent states:

null-recurrent: $\mu_{jj} = \infty$ (can only occur when I is infinite)

positive-recurrent: $\mu_{jj} < \infty$

If all states in I communicate they're all either transient or recurrent.

$$I_n = \begin{cases} 1 & X_n = j \\ 0 & X_n \neq j \end{cases} \qquad E\left[\sum_{k=1}^{n} I_k | X_0 = i\right] = \sum_{k=1}^{n} E[I_k | X_0 = i] = \sum_{k=1}^{n} P_{ij}^{(k)}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)} = \text{the long run avg. \# of visits to } j \text{ per unit of time when starting in } i$$

27/11 L10 Stochastic Processes

In a MCh w/ a finite I w/o two disjoint closed sets

- All states communicate
- All states are positive recurrent w/ same d

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)} = \frac{1}{\mu_{jj}} \quad \forall i,j \in I$$

If $d=1$: $\lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{\mu_{jj}}$

- $P\{X_n=j\} = \sum_{k\in I} P\{X_n=j \mid X_0=k\} P\{X_0=k\}$

$$\lim_{n\to\infty} P\{X_n=j\} = \sum_{k\in I} \lim p_{kj}^{(n)} P\{X_0=k\} = \frac{1}{\mu_{jj}} \sum_{k\in I} P\{X_0=k\} = \frac{1}{\mu_{jj}}$$

What if I is infinite?

Assume 3.3.1 $\exists r\in I$ w/ $f_{ir}=1$ for any $i\in I$ and $\mu_{rr}<\infty$

[There exists a positive reccurent state r which is accessible from any state $i \in I$]

Note: The assumption is true for a MCh. w/ finite I w/o two disjoint closed sets

3.3.2 The equilibrium equation

Def 3.3.2 A prob. distr. $\{\pi_j, j\in I\}$ is said to be an equilibrium for a MCh. w/ state space I and ~~the~~ $P=[P_{ij}]_{i,j\in I}$ for any $j \in I$ $\boxed{\pi_j = \sum_{k\in I} \pi_k P_{kj}}$

Expl.

Suppose $P\{X_0=j\}=\pi_j$ $j\in I$

$$P\{X_1=j\} = \sum_{k\in I} \underbrace{P\{X_1=j \mid X_0=k\}}_{P_{kj}} \underbrace{P\{X_0=k\}}_{\pi_k} = \sum_{k\in I} \pi_k \cdot P_{kj} = \pi_j \quad j\in I$$

Thus X_1 has distr. $\{\pi_j, j\in I\}$ by induction: the distr. of X_n is $\{\pi_j, j\in I\}$ $\forall n$.

Thm 3.3.2

Suppose assump. 3.3.1 holds for the MCh $\{X_n, n\geq 0\}$, then there is a unique equilibrium distribution $\{\pi_j, j\in I\}$ satisfying

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)} = \pi_j \ \forall k\in I$$

Any solution $\{x_j, j \in I\}$ of the equilibrium equations

$$x_j = \sum_{k \in I} x_k P_{kj} \quad \text{satisfies} \quad x_j = c\,\pi_j \quad \forall j \in I$$

$\{\pi_j, j \in I\}$ is found from

$$\pi_j = \sum_{k \in I} \pi_k P_{kj} \quad \sum_{j \in I} \pi_j = 1 \quad \text{(only one of these eqn. is needed).}$$
$$j \in I$$

~~Interpretation~~

Corollary 3.5.1 (p. 129)

If in thm 3.3.2 the state r is a-periodic then

$$\lim P_{ij}^{(n)} = \pi_j, \ \forall i, j \in I$$

Interpretation of the equilibrium prob

[Thm 3.3.1 (a)]

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P_{jj}^{(k)} = \begin{cases} \frac{1}{\mu_{jj}} & j \text{ reccurent} \\ 0 & \text{otherwise} \end{cases}$$

π_j - the long run average # of visits to j per time unit when starting in j.

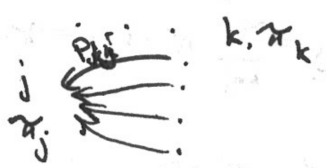
[Thm 3.3.2] Assumption 3.3.1

π_j = the number of visits to j per time unit.

r - aperiodic (ass. 3.3.1)

$$\pi_j = \lim_{n \to \infty} P_{ij}^{(n)} = \lim_{n \to \infty} P\{X_n = j\}$$

Physical interpretation:



Based on: in the long run the # of transitions to j = # transitions from j, $j \in I$

$$\pi_j = \sum_{k \in I} \pi_k \cdot P_{kj}$$

Ex: 3.2.1 [Continued]

$\{X_n, n = 0, 1, \ldots\}$ $I = \{0, 1, 2\}$ $\quad P = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.5 & 0.25 & 0.25 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$

I is finite, no two disjoint closed states. By thm 3.3.2

$\exists$ a unique equilibrium distribution $\quad \pi = [\pi_0, \pi_1, \pi_2]$, $\pi = \pi \cdot P$

$$\pi = [0.596, 0.1722, 0.2318]$$

all states aperiodic, thus $P_{ij}^{(n)} \xrightarrow[n \to \infty]{} \pi_j$ (obtained from before)

Ex 3.1.2 Contin.

A stock control problem w/ (s,S) inventory

X_n = the # of pliers on hand in week n prior to review

$X_0 = S$, Control at the beginning of each week (Control rule (s,S))

Weekly demands: Indep $Poi(\lambda)$ r.v.'s

$\{X_n, n \geq 0\}$ $X_0 = S$ $I = \{0, 1, \dots, S\}$ is a MCh. We have found P

$0 \dots i \dots s \dots j \dots S$

$$P_{sl} > 0 \quad l \in I$$
$$P_{js}^{(2)} > 0 \quad s \leq j \leq S-1$$
$$P_{is} > 0 \quad 0 \leq i < S$$

$C \subset I$, $l \in C \Rightarrow s \in C \Rightarrow j \in I$ is from C

The equilibrium distribution exists! $\{\pi_j, j \in I\}$ - the equib. dist.

The long run average freq. of ordering [the long run avg. # of ordering per week] = The long run avg. # of ~~weeks~~ visits to $\{0, 1, \dots, s-1\}$ per unit time [a week]

$$= \sum_{j=0}^{s-1} \pi_j$$

since $P_{ss} > 0$ S_e is aperiodic $\pi_j = \lim_{n \to \infty} P_{ij}^{(n)} = [\lim P\{X_n = j\}]$

The long run avg. stock / week $= \sum_{j \in I} j \cdot \pi_j$

The failure rate function

{App. A p. 438}

$X \geq 0$, $X \sim F(x)$; $X \sim f(x)$ (think of a life time)

The failure rate function [hazard rate, dead intensity] is defined as

$$* \quad r(t) = \frac{f(t)}{1 - F(t)} \quad \text{for } t \text{ s.t. } F(t) < 1$$

Interpretation: Choose t s.t. $P\{X > t\} > 0$ $[\Leftrightarrow F(t) < 1]$

consider,
$$P\{X < t+h \mid X > t\} = \frac{P\{t < X < t+h\}}{P\{X > t\}} = \frac{F(t+h) - F(t)}{1 - F(t)}$$

h

$t \quad t+h$

$$\underset{h \text{ small}}{\approx} \frac{f(t) \cdot h}{1 - F(t)} = r(t) \cdot h$$

$P\{$the lifetime ends in an interval of length h after $t \mid X > t\}$

$\approx r(t) \cdot h$

the dead intensity

from $(*)$ $\underbrace{f(t)}_{-[1-F(t)]'} = r(t)[1 - F(t)]$ $\quad F(0) = 1$

$$1 - F(t) = \exp\left\{ - \int_{x=0}^{t} r(x)\,dx \right\} \quad \binom{*}{*}$$

Suppose $r(t) = \lambda$, from $\binom{*}{*}$ $1 - F(t) = e^{-\lambda t} \Rightarrow X \sim Exp(\lambda)$

3.3.3 The long run average reward per time unit

$\{X_n, n \geq 0\}$ a MCh. assumption 3.3.1 holds

Suppose a reward $f(j)$ is earned for any visit to j, $\forall j \in I$

Assumptions

(a) $\sum_{j \in I} |f(j)| \pi_j < \infty$

(b) For each initial state $X_0 = i$ the total reward earned until the first visit to the positive reccurrent state r from ass. 3.3.1, is finite w/ prob. 1.

Thm 3.3.3 (under above assumptions)

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = \sum_{j \in I} f(j) \pi_j$$

1/12 Stochastic processes

Ex: 5 (3.11)

- A player who wins on a given game will lose the next w/ prob. .75
- He pays \$1 for each game and receives \$2.5 for each game won.

(If he loses, the prob. in next game are equal).

Is the game fair?

Solution

$$X_n = \begin{cases} 1 & \text{if the player wins game } n \\ 0 & \text{otherwise} \end{cases} \quad - X_n \text{ a MCh.}$$

$$P = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} .5 & .5 \\ .75 & .25 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \end{array}$$

The equilibrium probabilities exist [I finite, no 2 disjoint closed sets] (ass. 3.3.1 holds)

$\pi = [\pi_0, \pi_1]$ $\quad \pi_0 + \pi_1 = 1$

$\pi = \pi \cdot P$ the eq. equation.

$$\begin{cases} .5\pi_0 + .75\pi_1 = \pi_0 \\ \pi_0 + \pi_1 = 1 \end{cases} \Rightarrow \begin{array}{l} \pi_1 = .4 \\ \pi_0 = .6 \end{array}$$

[X the win in the long run game]

$E[X] = 2.5 \cdot P\{\text{win}\} - 1 = 2.4 \cdot 0.4 - 1 = 0$ thus the game is fair

Pr. 3.12

A container w/ capacity of 4 m^3. The weekly produced waste is

Vol:	$0\,m^3$	$1\,m^3$	$2\,m^3$	$3\,m^3$
Prob:	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The removal of 1 m^3 excess costs \$30.

Control at end of each week. Waste is removed if more than 2 m^3.

Prices:

\$25 for the removal

\$5 for each m^3 of removed waste.

What is long run average cost/week?

X_n = the amount of waste at the end of week n. (prior to check).

$I = \{0, 1, 2, 3, 4\}$

X_n is a MCh. w

$$P = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 \\ 0 & \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Eq. prob. exists $\pi = [\pi_0, \pi_1 \dots \pi_4]$

$$\sum \pi_i = 1 \qquad \bar{\pi}_0 = 1 \text{ find } \bar{\pi}_1, \dots, \bar{\pi}_4 \text{ which solve eqn.}$$
$$\pi = \pi \cdot P$$

$$\pi_j = \frac{\bar{\pi}_j}{\sum_{k \in I} \bar{\pi}_k}$$

$\cancel{\pi_0 = \frac{1}{2} + \frac{1}{8}\pi_1 + \frac{1}{\ }}$

$$\pi_0 = \frac{1}{8} + \frac{1}{8}(\bar{\pi}_3 + \bar{\pi}_4) \iff \pi_3 + \pi_4 - +$$

$$\pi_1 = \frac{1}{2} + \frac{1}{8}\bar{\pi}_1 + \frac{1}{2}\underbrace{(\bar{\pi}_3 + \bar{\pi}_4)}_{=7} \Rightarrow \bar{\pi}_1 = \frac{32}{7}$$

$$\pi_2 = \frac{1}{4} + \frac{1}{2}\bar{\pi}_1 + \frac{1}{8}\bar{\pi}_2 + \frac{1}{4}\underbrace{(\bar{\pi}_3 + \bar{\pi}_4)}_{7} \quad \bar{\pi}_2 = \frac{240}{49}$$

$$\bar{\pi}_4 = \frac{118}{29}$$

$$\sum \bar{\pi}_j = \frac{856}{49} \Rightarrow \begin{aligned} \pi_0 &= 0.05 \\ \pi_1 &= 0.262 \\ \pi_2 &= 0.280 \\ \pi_4 &= 0.138 \\ \pi_3 &= 0.263 \end{aligned}$$

Cost = direct removal cost + the expected cost for excess removal. next week.

$f(0) = f(1) = 0$

$f(2) = 30 \cdot ($[redacted]$ + \frac{1}{8})$

$f(3) = 25 + 5 \cdot 3 = 40$

$f(4) = 25 + 5 \cdot 4 = 45$

The long run average cost/week $= \sum_{j \in I} f(j) \cdot \pi_j = 17.78$

__Example__

On average 230 bottles of beer are stored in the inventory with 100 (on average) sold/week

W = the average # of weeks spent in the inventory.

$230 = 100 \cdot W$ Littles formula (w/ money principle).

__Infinite capacity systems__

We assume that every arriving customer is allowed to wait to stay in the system until service is provided.

$L_q(t)$ = # of customers in the queue at time t (excluding those in service).

$L(t)$ = # of customers in the system at time t.

D_n = the amount of time spent by the n-th customer in queue (excluding those in service)

U_n = the amount of time spent by the n-th customer in the system.

__Assumptions__ $\{L_q(t)\}\{L(t)\}, \{D_n\}, \{U_n\}$ are regenerative with a finite expectation of the cycle.

Then $\exists$: constants L_q, L, W_q, W s.t.

$$\lim_{t \to \infty} \frac{1}{t}\int_0^t L_q(u)\,du = L_q \qquad \lim_{t \to \infty} \frac{1}{t}\int_0^t L(u)\,du = L$$

$$\lim_{n \to \infty} \frac{1}{n}\sum_{j=1}^n D_j = W_q \qquad \lim_{n \to \infty} \frac{1}{n}\sum_{j=1}^n U_j = W$$

system delay

- $A(t)$ = the number of customers which have arrived at time t.

assume
$$\lim_{t\to\infty} \frac{A(t)}{t} = \lambda \quad \text{the rate [avg. \# of cust / unit of time]}$$

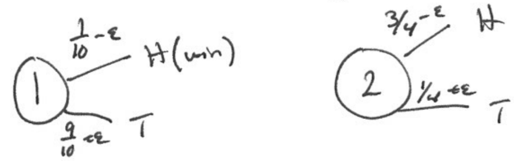
Little's formula

$$L_q = \lambda W_q$$
$$L = \lambda W$$

Pr. 3.13

- Peter plays a game $\begin{cases} \text{win} \to €1 \\ \text{loss} \to -€1 \end{cases}$ H - win, T - loss.

- X_n = # of € after game n.



$X_n = 3Z_n + Y_n$ —— remainder $Y_n = \{0,1,2\}$

Z_n: the greatest integer s.t. ~~$3Z_n \le X_n$~~ $3Z_n \le X_n$

If $Y_n = 0$ P uses coin 1 in next game

otherwise coin 2.

What is t. long run fraction of ~~time~~ games won by Peter?

$\{Y_n\}$ is a MCh w. $I = \{0,1,2\}$

and
$$P = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{10}-\varepsilon & \frac{9}{10}+\varepsilon \\ \frac{1}{4}+\varepsilon & 0 & \frac{3}{4}-\varepsilon \\ \frac{3}{4}\cancel{+}\varepsilon & \frac{1}{4}+\varepsilon & \frac{3}{4}-\varepsilon \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \end{array}$$

$\pi = [\pi_0, \pi_1, \pi_2]$
$$\begin{cases} \pi = \pi P \\ \sum \pi_i = 1 \end{cases}$$

$\varepsilon = 0.005$
$\pi_0 = 0.3836$
$\pi_1 = 0.1343$
$\pi_2 = 0.4622$

in the long run Peter uses 1 w. prob. π_0 and coin 2 w/ prob. $(\pi_1 + \pi_2)$

$$Z = \begin{cases} 1 & \text{if Peter wins} \\ 0 & \text{otherwise} \end{cases}$$

$$E[Z] = E[Z|1] \cdot \pi_0 + E[Z|2](\pi_1+\pi_2)$$
$$= 1\cdot\left(\frac{1}{10}-\varepsilon\right)\cdot\pi_0 + 1\left(\frac{3}{4}-\varepsilon\right)(\pi_1+\pi_2)$$

long run avg. ~~# of~~ fraction of games won.

$$= 0.4957$$

3/12 Stochastic processes. L11

Ch 4: Continuous time MCh.

Def 4.1:
A continuous time MCh is a Markov process with
- continuous time
- discrete state space

$\{X(t), t \geq 0\}$ a MCh. w/ state space I.

Recall the Markov property

$n \geq 2$ $\quad P\{X(t_{n+1}) = i_{n+1} \mid X(t_n) = i_n, \ldots, X(t_1) = i_1\}$

$= P\{X(t_{n+1}) = i_{n+1} \mid X(t_n) = i_n\}$

$t_1 \quad t_2 \quad t_n \quad t_{n+1}$ — past, present, future

The probabalistic future depends on
- the present state
- possibly on t_n

but not on the time spent in i_n before t_n (during the present visit)

- Thus the visit times are memoryless, hence exponential.

Our model

$\{X(t), t \geq 0\}, I$

a) The visit time in $i \in I$ has exponential distribution w/ mean $\frac{1}{\nu_i}$, independently of the process prior to the visit.

(b) From state $i \in I$ the process jumps to $j \in I$ w/ probability P_{ij} where $\sum_{\substack{j \in I \\ j \neq i}} P_{ij} = 1$

(c) The # of jumps in a finite time interval is finite w/ probability 1.

Comments

(b) self visits are not allowed. Is it a restriction?

Assume $P_{ii} > 0$

M = # of self visits before departure

$i \xrightarrow{P_{ii}}$ $\quad 1 - P_{ii}$

$P\{M = m\} = \underbrace{P_{ii}^m (1 - P_{ii})}_{\text{Geom}(1-P_{ii})}, \; m = 0, 1, \ldots$

T = total visit time

$T = \underset{\text{first visit}}{X_0} + \underset{\text{first self visit}}{X_1} + \ldots + X_M = \sum_{k=0}^{M} X_k \quad X_k \sim \text{Exp}(\nu_i)$, indep.

$$P\{T > t\} = \sum_{m=0}^{\infty} P\left\{\sum_{k=0}^{M} X_k > t \mid M = m\right\} P\{M = m\}$$

$$= \sum_{m=0}^{\infty} \underbrace{P\{S_{m+1} > t\}}_{P\{N(t) \leq m\}} P\{M = m\} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(\nu_i t)^k}{k!} e^{-\nu_i t} P_{ii}^m (1 - P_{ii})$$

$X_0 \quad X_1 \quad \cdots \quad X_M \quad S_{m+1}$

$m+1$:st arrival of a Poisson process w/ parameter ν_i

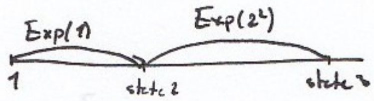
$= e^{-\nu_i(1-P_{ii})t}$ $T \sim Exp(\nu_i(1-P_{ii}))$

We can then exclude self visits and assume the visit time to $i \in I$ is $Exp(\nu_i(1-P_{ii}))$

(c) <u>Example</u>

$I = \{1, 2, \dots\}$ $X_i \sim Exp(i^2)$ the visit time in state i.

$X(0) = 1$, $P_{i,i+1} = 1$



Average time for N jumps

$$= \sum_{i=1}^{N} \frac{1}{i^2} < \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

$P\{\text{inf. many jumps in } [0, \frac{\pi^2}{6}]\} > 0$!

<u>Note:</u>

The model is defined by $\{\nu_i, P_{ij}, i,j \in I; i \neq j\}$

<u>Example 4.1.4</u>

Customers arrive as a PP w/ rate λ. Each customer requests 1 unit of product

There are Q units initially.

- Control according to a PP w/ rate μ (processes independent)

Replenishment is only done when the tank is empty.

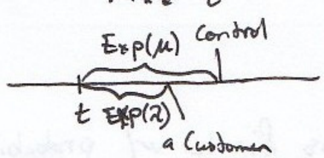
<u>Question</u>: What is the long run average of stock in the tank?

What is the long run fraction of ~~the~~ demand which is lost?

The model: $X(t)$ = the amount of stock at time t, $t \geq 0$

$I = \{0, 1, \dots, Q\}$

Fix t



Suppose $X(t) = i$, the sojourn time in i is

$Exp(\nu_i)$, $\nu_i = \begin{cases} \lambda & i = 1, 2, \dots, Q \\ \mu & i = 0 \end{cases}$

$P_{i,i-1} = 1$, $i = 1, 2, \dots, Q$

$P_{0,Q} = 1$

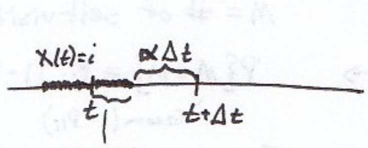
<u>Infinitesimal transition</u>

$\{X(t), t \geq 0\}$, I is a MCh.

Fix $t \geq 0$ $i \in I$

Let $j \neq i$

$\alpha \in (0,1)$



$P\{X(t+\Delta t) = j \mid X(t) = i\} =$

$= P\{X(t+\Delta t) = j, \text{ 1 jump in } \Delta t \mid X(t) = i\} + P\{X(t+\Delta t) = j, \geq 2 \text{ jumps in } \Delta t \mid X(t) = i\}$

$\approx \underbrace{\nu_i \Delta t P_{ij}}_{\text{jump to } j \text{ within } \Delta t} \cdot \underbrace{e^{-\alpha \Delta t \nu_j}}_{P\{\text{st. stays in } j \text{ up to } t+\Delta t\}} + O(\Delta t)$

infinitesimal w/ respect to Δt

$$\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0.$$

$$\lim_{\Delta t \to 0} \frac{P\{X(t+\Delta t) = j \mid X(t) = i\}}{\Delta t} = \underbrace{\nu_i P_{ij}}_{\text{transition rate for jumps from } i \text{ to } j.}$$

$$* \boxed{\begin{array}{l} q_{ij} = \nu_i P_{ij} \quad i,j \in I \quad i \neq j \\ \text{the infinitesimal transition rate} \end{array}}$$

<u>Let $i=j$</u>

$$P\{X(t+\Delta t) = i \mid X(t) = i\} = 1 - \nu_i \Delta t + \Theta(\Delta t)$$

$$P\{X(t+\Delta t) = i \mid X(t) = i\} = \begin{cases} \approx q_{ij} \Delta t & j \neq i \\ 1 - \nu_i \Delta t & j = i \end{cases}$$

Given $\{q_{ij},\ i,j \in I\ i \neq j\}$ the set $\{\nu_i, P_{ij}, i,j \in I\ i \neq j\}$

is uniquely defined

In $(*)$ take $\sum\limits_{\substack{j \in I \\ i \neq j}} q_{ij} = \nu_i \sum\limits_{\substack{j \in I \\ j \neq i}} P_{ij} = \nu_i \quad , \quad P_{ij} = \frac{q_{ij}}{\nu_i}$

We assume $q_{ij} > 0 \ , \ \overbrace{\sum\limits_{\substack{j \in I \\ j \neq i}} q_{ij}}^{\nu_i} < M$ a number

under this condition $\{q_{ij}\}$ defines our model uniquely.

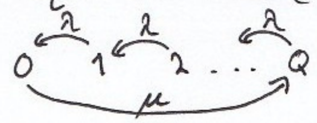
$\frac{1}{\nu_i} > \frac{1}{M}$ We exclude jump explosions

<u>Example 4.1.1</u>

$\{X(t), t \geq 0\}$ the amount of stock $I = \{0, 1, \dots, Q\}$

$i \in [1, Q]$ $P\{X(t+\Delta t) = i-1 \mid X(t) = i\} = P\{\text{demand in } (t, t+\Delta t) \mid X(t) = i\}$

$\approx \lambda \Delta t$



— transition rate diagram of cont. MCh.

<u>Pr. 4.1</u>

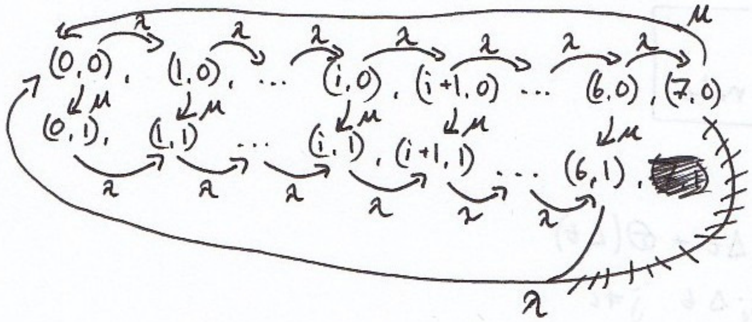
- A sheroot has room for 7 passengers and leaves from a fixed stop in town ~~for~~ a different town
- It leaves as soon as all seats are filled, The stop has only room for one sheroot.
- When a sheroot leaves a new one arrives after an exp time w/ mean $1/\mu$.
- Passengers arrive according to a P.P. w/ rate λ. When a customer finds 7 or more passengers waiting for a sheroot the customer goes somewhere else.

<u>The model:</u>

$X_1(t)$ = # of waiting people at time t.

$X_2(t) = \begin{cases} 1 & \text{if a sheroot is present at time } t \\ 0 & \text{otherwise} \end{cases}$

$X(t) = X_1(t), X_2(t)$ $\qquad \{X(t), t \geq 0\}$ is a MCh. w/ $I = \{(i,0),\ i \in [0,7]\} \cup \{(1,1),\ i \in [0,6]\}$



$$P\{X_1(t+\Delta t)=i+1,\ X_2(t+\Delta t)=0 \mid X_1(t)=i,\ X_2(t)=0\}$$

$$= P\left\{\begin{array}{l}1 \text{ passenger in } (t,t+\Delta t)\\ 0 \text{ sherouts in } (t,t+\Delta t)\end{array}\right\}$$

$$\approx \lambda\cdot\Delta t\,[1-\mu\cdot\Delta t]$$

$$= \lambda\Delta t + O(\Delta t)$$

4/12 Stochastic processes

$$I_j(u) = \begin{cases} 1 & X(u)=j \\ 0 & \text{otherwise} \end{cases}$$

$$\underbrace{E[\, I_j(t+\Delta t) \mid X(t)=i \,]}_{\{\text{avg. \# of jumps to } j \text{ in time } \Delta t \mid \text{starting in state } i\}} = 1 \cdot P\{X(t+\Delta t)=j \mid X(t)=i\} \approx q_{ij} \cdot \Delta t + O(\Delta t)$$

L 12

4.2 The flow rate equation method

$\{X(t), t\geq 0\}$ a cont. time MCh. w/ state space I & infinitesimal trans rates $\{q_{ij},\ i,j\in I,\ i\neq j\}$

[Assumption 4.1.2 holds] $\Leftrightarrow$ (no jump explosions in finite time intervals)

Define the transient probabilities

$$P_{ij}(t) = P\{X(t)=j \mid X(0)=i\} \quad \forall i,j \in I \qquad \text{(defined by D.E)}$$

Questions of interest

- Is there any $\lim\limits_{t\to\infty} P_{ij}(t) = P_j$ independent of $i\in I$?
- how is P_j related to the long run fraction of time the process is in state j?
- define the first visit times $k\in I: \tau_k = \min\{t>0: X(t)=k\}$ (first visit to k after $t=0$)

Assumption 4.2.1

$\{X(t), t\geq 0\}$ has a regenerative state r, s.t. for any $i\in I$

$$P\{\tau_r<\infty \mid X(0)=i\}=1 \qquad \text{(probability to ever reach } r \text{ from } i=1)$$

$$E[\tau_r \mid X(0)=r]<\infty \qquad \text{(the mean recurrence time is finite)}$$

Def: $\mathbb{I}_j(t) = \begin{cases} 1 & \text{if } X(t)=j \\ 0 & \text{otherwise} \end{cases}$

$\frac{1}{t}\int_0^t \mathbb{I}(u)\,du =$ the proportion of time spent in j up to time t.

Under the assumption 4.1.2 for $j\in I$

$$\lim_{t\to\infty} P_{ij}(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t I_j(u)\,du = p_j \quad \forall i\in I.$$

Interpretation of P_j

$\{P_j, j \in I\}$ is the equilibrium distribution. [An outsider who visits the system described, at a time long after 0, we will find the system in state j w/ probability $P_j, j \in I$].

- P_j - the long run amount of time spent in j per unit of time.
- Suppose visitors arrive according to a PP $N(t)$
  Assume: future visitors (for any $u \geq 0$) $\{N(t), t>0\}$ and $\{X(t), 0 \leq t \leq u\}$ are independent [lack of anticipation property].

  Then in the long run
  $P\{$a visitors finds $X(t)$ in state $j\} \approx P_j \quad j \in I$ [PASTA!].

Computation of $\{P_j, j \in I\}$

Thm. 4.2.1

Suppose the continuous time MCh satisfies assumption 4.1.2 (no jump explosions) and 4.2.1 (ensuring existence of $\{P_j, j \in I\}$) then $\{P_j, j \in I\}$ are the solutions (unique) to

$$(*) \quad \nu_j x_j = \sum_{k \neq j} q_{kj} x_k \quad j \in I \qquad \text{satisfying} \sum_{k \in I} x_k = 1$$

the balance equations

normalising.

Physical interpretation of (*)

LRANT = long run avg. # of transitions

$A \subset I$

LRANT from A per unit of time = LRANT into A per unit of time (the flow rate principle).

P_k = the long run amount of time spent in k per unit of time.

q_{kj} = the average number of transitions to j per unit of time, starting in k.

$q_{kj} P_k$ = the long run avg. # of transitions into j ~~from~~ per unit of time from k.

$\sum_{\substack{k \in I \\ k \neq j}} q_{kj} P_k$ = the long run number # of transitions into j per unit of time.

$\underbrace{\sum_{\substack{i \in I \\ i \neq j}} q_{ji} P_j}$ = LRANT from j per unit of time

$P_j \underbrace{\sum_{\substack{i \neq j \\ i \in I}} q_{ji}}_{\nu_j} = P_j \cdot \nu_j$ By the flow rate principle $\nu_j P_j = \sum_{\substack{k \neq j \\ k \in I}} q_{kj} P_k$

ACI

$$\underbrace{\sum_{j\in A} p_j \sum_{k\in \bar{A}} q_{jk}}_{\text{LRANT from } A.} = \underbrace{\sum_{i\in \bar{A}} p_i \sum_{j\in A} q_{ij}}_{\text{LRANT into } A.}$$

One particular case:

$i-1 \quad i$

$q_{ij} = 0, \; j \le i-2$

$A = \{i, i+1, \dots, N\} \quad i \ge 1$

$I = \{0, 1, 2, \dots, N\}$ (N may be infinite)

$i=1$:

$\underbrace{0, 1, 2, 3, \dots}$

$\underbrace{p_1}_{\text{known}} q_{10} = p_0 \underbrace{\sum_{j=1}^{N} q_{0j}}_{\text{known}}$ $\qquad \bar{p}_0$; find $\bar{p}_1$

$i=2$:

$\underbrace{0, 1, 2, 3, \dots}$

$p_2 q_{21} = \underset{\bar{p}_0}{p_0} \underbrace{\sum_{j=2}^{N} q_{0j}}_{\text{known}} + \underset{\bar{p}_1}{p_1} \underbrace{\sum_{j=2}^{N} q_{1j}}_{\text{known}}$ $\qquad$ obtain $\bar{p}_2$

$\vdots$

We obtain a recursion scheme for computing $\bar{p}_j$: $\quad p_j = \dfrac{\bar{p}_j}{\sum_{i\in I} \bar{p}_i}$

Continuous time MCh w/ rewards

- $\{X(t), t \ge 0\}$; I; $\{p_j, j\in I\}$
- Assume a reward of rate $r(j)$ for being in state j.
- $\sum_{j\in I} |r(j)| p_j$
- Assume the total reward until the first visit to the recurrent state r is finite, whatever the initial state at $t=0$.
- $R(t)$ = the reward accumulated up to time t.

Thm: 4.2.2

under the above assumptions

$$\lim_{t\to\infty} \frac{R(t)}{t} = \sum_{j\in I} r(j) p_j$$

Ex. 4.1.1 continued

$I = \{0, 1, \dots, Q\}$

Q is a recurrent state. Equilibrium probabilities exist

$i=0$

$p_0 \mu = p_1 \lambda$: $p_1 = \frac{\mu}{\lambda} p_0 = p_2 = \dots = p_Q$

$1 \le i \le Q-1$

$\lambda p_i = \lambda p_{i+1}$

$$\sum_{j=0}^{Q} p_j = p_0\left[\frac{\mu}{\lambda}(Q) + 1\right] = p_0\left[1 + \frac{\mu}{\lambda} Q\right]$$

$i = Q$

$p_Q \lambda = \mu p_0$

thus $1 = p_0\left(1 + \frac{\mu}{\lambda} Q\right) \Leftrightarrow p_0 = \dfrac{1}{1 + \frac{\mu}{\lambda} Q} = \dfrac{\lambda}{\lambda + \mu Q}$

$$p_i = \frac{\mu}{\lambda} \cdot \frac{\lambda}{\lambda + Q\mu} = \frac{\mu}{\lambda + Q\mu} \qquad 1 \le i \le Q$$

The long run average stock on hand is defined as

$\lim_{t \to \infty} \frac{1}{t} \int_0^t X(u)\,du$ – the long run average amount of stock / time unit.

assume a reward of rate j for being in j [j = reward for being in j 1 unit of time]

By Thm 4.2.2

long run avg. stock on hand $= \sum_{j=0}^{Q} j\, p_j = \sum_{j=1}^{Q} j \cdot \frac{\mu}{\lambda + Q\mu} = \frac{\mu}{\lambda + Q\mu} \frac{Q(Q+1)}{2}$

- The long run fraction of demands lost $= p_0$ [the long run fraction of time a Poisson visitor finds the system in state 0].

- The long run average number of replenishments per time unit $= p_0 \cdot \mu$

Pr. 4.4

- Cars arrive according to a PP w/ rate 10 cars/h
- A car enters the station only if less than 5 other cars are present.
- The station has one pump w/ exp. w/ mean 4 minutes

(a) Formulate the model

(b) What is long run average # of cars in the station?

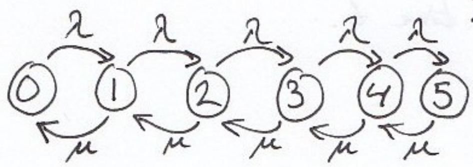
(c) What is long run fraction of lost customers?

$X(t)$ = # of cars at the station at time t

$I = \{0, 1, 2, 3, 4, 5\}$ time unit 1h.

$\lambda = 10/h$

$\frac{1}{\mu} = \frac{4}{60}$, $\mu = 15$ services/h.



8/12 Stochastic Processes.

Pr. 4.4 cont.

(b) The equilibrium distribution exists

For $i=1,\dots,5$ let $A_i=\{i,\dots,5\}$

By the flow rate equation method we have

$$\begin{cases} \lambda p_{i-1} = \mu p_i & 1\le i\le 5 \\ \sum_{i=0}^{5} p_i = 1 & \text{normalising eqn.} \end{cases}$$

$\bar{p}_0 = 1$:

$\bar{p}_1 = \frac{\lambda}{\mu} = \rho$: $\lambda=10,\ \mu=15,\ \rho=\frac{2}{3}<1$!

$\bar{p}_2 = \rho p_1 = \rho^2$

$\bar{p}_5 = \rho^5$

$$\sum_{i=0}^{5} \rho^i = \frac{1-\rho^6}{1-\rho} \ : \quad p_i = \frac{\bar{p}_i}{\frac{1-\rho^6}{1-\rho}} = \frac{\rho^i(1-\rho)}{1-\rho^6}$$

$p_0 = .365\dots$
$p_1 = .243\dots$
$p_2 = .162\dots$
$p_3 = \dots$
$p_4 = \dots$
$p_5 = .048\dots$

The long run average # of cars at the station per time unit,

$= \sum_{i=1}^{5} i p_i = 1.423$

(c) The long run fraction of lost customers is $p_5 = .048$

The long run avg. # rate of rejected cars per unit of time $= \lambda p_5$

TQ 5.5.1 The M/M/1 queue

M – the exponential distr.

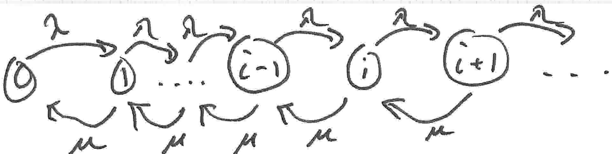
- One server in the system
- Customers arrive according to a PP. of rate λ
- The service times are indep and exponential w/ mean $\frac{1}{\mu}$; it is assumed that $\mu > \lambda$
- If an arriving customer finds the server free, the customer enters service immediately, otherwise the customer joins the queue.

Of interest: the long run behaviour of the system:

(a) the model

$X(t)$ = # of customers at time t.

$\{X(t), t\ge 0\}$ is a contin. time MCh w. $I=\{0,1,\dots\}$



$\lambda \cdot \frac{1}{\mu} = \rho$ - average amount of work offered to the server per unit of time.

If $\rho < 1$ (= the server capacity / time unit)
then the system will return to state 0 over and over again.
Assumption 4.2.1 holds and an equilibrium distribution exists.

Computation of $\{P_j, j \in I\}$
$A_i = \{i, i+1, \dots\}$

$$\lambda p_{i-1} = \mu p_i \quad i = 1, 2, \dots$$

$$\bar{p}_0 = 1, \ \bar{p}_1 = \rho^2 \ \cdots \ \bar{p}_i = \rho^i \quad \sum_0^\infty \rho^i = \frac{1}{1-\rho}; \quad p_i = \rho^i \cdot (1-\rho) \quad i = 0, 1, \dots$$

L_q = the long run average # of customers in queue per time unit.

$$\overset{=}{\text{def}} \lim_{t \to \infty} \frac{1}{t} \int_0^t [X(u) - 1]^+ du \underset{\text{thm. 4.2.2}}{\Longrightarrow} \sum_{i=\cancel{1}2}^\infty (i-1) p_i = \rho^2 (1-\rho) \sum_{i=2}^\infty (i-1) \rho^{i-2}$$

$$= \rho^2 (1-\rho) \sum_{j=1}^\infty j \rho^{j-1} = \rho^2 (1-\rho) \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) = \frac{\rho^2 (1-\rho)}{(1-\rho)^2}$$

The long-run fraction of customers who find j other customers upon arrival $= P_j$ by the PASTA property.

Pr. 4.8 |

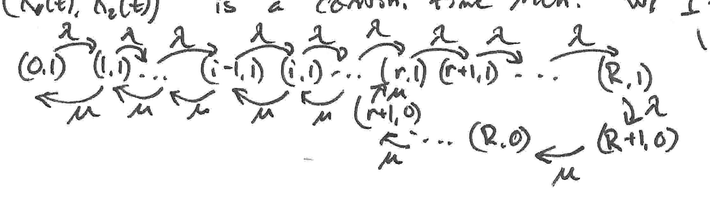
Messages arrive at a transmission chan~~l~~ as a ~~P~~ PP w/ rate λ. One message can be transmitted at a time and the transmission times are indep and exp w/ mean $\frac{1}{\mu}$.
A arrived message is accepted only if there are at most R other messages ~~at~~ the channel.
If there are $R+1$ or more messages at the channel the newly arrived messages and the following arrivals are rejected as long as the number of messages at the channel remains ~~greater than / or equal to $R+1$~~. $r+1$ or more
When ~~R~~ this number drops to r a newly arrived message ; accepted. $r < R$.

Q: (a) What is the long fraction of time the channel is idle?
(b) What is the long run avg # of messages rejected?
(c) What is the long run average # of messages L_q waiting for transmission
(d) What is the long-run average delay in queue per accepted message?

Sol:
$X_1(t)$ = # of messages at system at time t. [including a message in transmission]

$$X_2(t) = \begin{cases} 1 & \text{the gate is open} \\ 0 & \text{otherwise} \end{cases}$$

$X(t) = (X_1(t), X_2(t))$ is a contin. time MCh. w/ $I = \{(i,1), i \in [0,R]\} \cup \{(i,0), r+1 \le i \le R+1\}$
finite, $(0,1)$ positive recurrent state $\Rightarrow$ Equib. distr. exist



~~P(i,j)~~

$\{P[(i,j)],\ (i,j)\in I\}$ - the equilibrium prob.

(a) $p[(0,1)]$

(b) $\sum_{i=r+1}^{R+1} p(i,0)$

(c) $L_q = \sum_{i=1}^{R} (i-1)\, p[(i,1)] + \sum_{i=r+1}^{R+1} (i-1)\, p[(i,0)]$

(d) $W_q = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} D_k$ D_k - the time of k:th message ~~spent~~ at the channel. [in queue]

By Little's formula:

$L_q = W_q \cdot \lambda_{accepted}$, $\lambda_{accepted} = \lambda\cdot(1 - P_{rejected})$

$W_q = \dfrac{L_q}{\lambda(1-P_{rejected})}$