EXAM
MVE166/MMG631
Linear and integer optimization with applications

- **Date:** 2025-06-05
  - **Hours:** 08:30–12:30
- **Aids:** Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3) points
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade
  - Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
- **Examiner:** Ann-Brith Strömberg
  - **Phone:** 0705-273645

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

ILP MODELING

[3p]

A logistics company is expanding its operations and is considering opening new warehouses to improve service levels across the region. The company has identified seven candidate cities for potential warehouse locations: *Aldora, Bexhill, Crestville, Dunwick, Elbridge, Fairmere, and Glenrock.*

For each city $i \in \{\text{Aldora, Bexhill, Crestville, Dunwick, Elbridge, Fairmere, Glenrock}\}$ there is a fixed cost, a_i for opening a warehouse, a salary cost, s_i per employee/work day, and a maximum number, u_i of employees it can accommodate. Due to budget limitations, the company may open at most five warehouses.

The company must also serve ten customer zones $z_1, \dots, z_{10}$, each with a specified, estimated daily demand of d_j packages to be delivered. Each customer zone must be assigned to exactly one open warehouse.

The required number of employees at each warehouse depends on the total daily demand it handles, based on the rule that one (1) employee is needed per 100 packages/work day. It is assumed that employees can only be permanently employed and for full work days.

The transportation costs per package from candidate city i to customer zone z_j is c_{ij}.

The company wishes to minimize its daily costs for meeting the estimated demand of package deliveries.

Formulate an *Integer Linear Program* (ILP) that determines:

- Which cities should host warehouses?
- How many employees should be hired at each of the open warehouses?
- Which customer zones should be assigned to each warehouse?

You should not solve the problem.

Question 2

THE SIMPLEX ALGORITHM

Consider the linear optimization problem to

$$
\begin{array}{llrcrcrcl}
\text{maximize} & z = & & & x_2 & + & x_3, & & \\
\text{subject to} & & x_1 & + & 2x_2 & + & 2x_3 & \leq & 7, \\
& & 2x_1 & + & x_2 & + & x_3 & \leq & 6, \\
& & & & & & x_3 & \geq & 2, \\
& & x_1, & & x_2, & & x_3 & \geq & 0.
\end{array}
$$

(a) **[1p]**

Reformulate this problem such that it can be solved using the simplex method.

(b) **[1.5p]**

Solve the problem using the simplex method, including all necessary steps.

(c) **[0.5p]**

State all optimal bases as well as all optimal points.

Question 3

CONVEXITY OF THE FEASIBLE SET OF A LINEAR OPTIMIZATION PROBLEM

[3p]

Consider a linear optimization problem, stated as

$$
\begin{array}{lll}
\text{minimize} & z = \displaystyle\sum_{j=1}^{n} c_j x_j, & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i, & i = 1, \ldots, m, \\
& x_j \geq 0, & j = 1, \ldots, n.
\end{array}
$$

Theorem 4.1 in the course book expresses that the feasible set of this problem is a convex set. State and prove this theorem.

Question 4

Cutting planes

Consider the ILP to

$$\begin{aligned} \text{minimize} \quad & -2x_1 - x_2, && \text{(1a)} \\ \text{subject to} \quad & 6x_1 + 2x_2 \leq 19, && \text{(1b)} \\ & x_1 + 5x_2 \leq 20, && \text{(1c)} \\ & x_1, \; x_2 \in \mathbb{Z}_+. && \text{(1d)} \end{aligned}$$

(a) **[1p]**

State a cutting plane that can be introduced due to the property of the inequality (1b).

(b) **[2p]**

Assume that the simplex method finds the following optimal tableau for the LP-relaxation of the ILP, where the improved inequality from sub-question (a) is used:

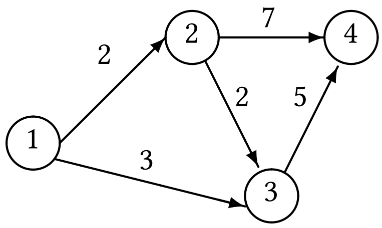
$\mathbf{x}_B$	z	x_1	x_2	s_1	s_2	$\mathbf{B}^{-1}\mathbf{b}$
z	1	0	0	$-9/14$	$-1/14$	$-7\frac{3}{14} = -101/14$
x_1	0	1	0	$5/14$	$-1/14$	$1\frac{11}{14} = 25/14$
x_2	0	0	1	$-1/14$	$3/14$	$3\frac{9}{14} = 51/14$

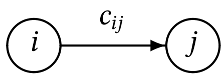
State all Gomory cuts that can be derived from the constraints in this tableau.

Question 5

Shortest paths and sensitivity analysis

Consider the following network of four nodes and five directed arcs with lengths c_{ij}, where (i, j) denotes the arc from node i to node j:





(a) **[1p]**

Solve the problem of finding a shortest path from node 1 to node 4 in this graph. Use a suitable method from the course, and motivate why this method is suitable for the data in this specific instance. Report your calculations clearly.

(b) **[1p]**

The problem of finding the shortest paths *from one node to all other nodes* in a directed graph can be formulated as a linear optimization problem. For the specific graph above the problem of finding the shortest paths from node 1 to each of nodes 2, 3, and 4 is formulated as to

$$\begin{array}{llrrrrrrrrrrcrr}
\text{minimize} & z = & & 2x_{12} & + & 3x_{13} & + & 2x_{23} & + & 7x_{24} & + & 5x_{34}, & & & \text{(2a)} \\
\text{subject to} & & - & x_{12} & - & x_{13} & & & & & & & = & -3, & \text{(2b)} \\
 & & + & x_{12} & & & - & x_{23} & - & x_{24} & & & = & 1, & \text{(2c)} \\
 & & & & + & x_{13} & + & x_{23} & & & - & x_{34} & = & 1, & \text{(2d)} \\
 & & & & & & & & + & x_{24} & + & x_{34} & = & 1, & \text{(2e)} \\
 & & & x_{12}, & & x_{13}, & & x_{23}, & & x_{24}, & & x_{34} & \geq & 0. & \text{(2f)}
\end{array}$$

The optimal solution to (2) is given by $x_{12} = 1$, $x_{13} = 2$, $x_{23} = 0$, $x_{24} = 0$, $x_{34} = 1$.

State the corresponding optimal basis to the problem (2a)–(2d), (2f). State also the three shortest paths, in terms of the arcs included in each path.

Note that the equations (2b)–(2e) *are linearly dependent. Therefore, one of these constraints, e.g.,* (2e)*, must be removed before a basis can be determined.*

(c) **[1p]**

Assume that the arc length $c_{13} = 3$ is replaced by $c_{13} = 3 + \delta_{13}$ and that the arc length $c_{23} = 2$ is replaced by $c_{23} = 2 + \delta_{23}$, where $\delta_{13}, \delta_{23} \in \mathbb{R}$.

The current solution to the problem (2a)–(2d), (2f) (i.e., the optimal solution given in sub-question (b) above) is optimal when the reduced costs of all non-basic variables are non-negative, i.e., when the inequality $\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^\top$ holds. Utilize this inequality to determine for which values of $(\delta_{13}, \delta_{23}) \in \mathbb{R}^2$ the current solution is optimal.

Solution proposals to EXAM 2025-06-05

MVE166/MMG631 Linear and integer optimization with applications

These solutions may be brief in relation to the requirements on your answers, in particular regarding motivations.

Solutions to Question 1

Decision Variables

- $x_i \in \{0,1\}$: 1 if a warehouse is opened in city i, 0 otherwise
- $y_{ij} \in \{0,1\}$: 1 if customer zone j is assigned to warehouse i, 0 otherwise
- $e_i \in \mathbb{Z}_{\geq 0}$: Number of employees hired in warehouse i

$$\text{ILP:} \qquad \min \quad \sum_{i=1}^{7} s_i e_i + \sum_{i=1}^{7}\sum_{j=1}^{10} c_{ij} d_j y_{ij}$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{i=1}^{7} x_i \leq 5 && \text{(At most 5 warehouses)} \\
& \sum_{i=1}^{7} y_{ij} = 1 && \forall j \in \{1,\dots,10\} \quad \text{(Each zone assigned to one warehouse)} \\
& y_{ij} \leq x_i && \forall i,j \quad \text{(Zones assigned only to open warehouses)} \\
& \frac{1}{100}\sum_{j=1}^{10} d_j y_{ij} \leq e_i && \forall i \quad \text{(Sufficient \# employees per warehouse)} \\
& e_i \leq u_i x_i && \forall i \quad \text{(Maximum \# employees per warehouse)} \\
& x_i, y_{ij}, e_i \in \mathbb{Z}_+ && \forall i,j
\end{aligned}$$

Solutions to Question 2

(a) Add slack variables to express the model on standard form:

$$\begin{array}{lrcrcrcrcrcrcl}
\text{maximize} & z = & & & & x_2 & + & x_3, & & & & & & \\
\text{subject to} & & & x_1 & + & 2x_2 & + & 2x_3 & + & s_1 & & & & = 7, \\
& & & 2x_1 & + & x_2 & + & x_3 & & & + & s_2 & & = 6, \\
& & & & & & & x_3 & & & & & - \; s_3 & = 2, \\
& & & x_1, & & x_2, & & x_3, & & s_1, & & s_2, & s_3 & \geq 0.
\end{array}$$

Since the column for s_3 is a negative unit column, an artificial variable and the two-phase method is required to solve the problem. The phase-I problem is stated as to

$$\begin{array}{lrcrcrcrcrcrcrcl}
\text{minimize} & w = & & & & & & & & & & & & a, & \\
\text{subject to} & & & x_1 & + & 2x_2 & + & 2x_3 & + & s_1 & & & & & = 7, \\
& & & 2x_1 & + & x_2 & + & x_3 & & & + & s_2 & & & = 6, \\
& & & & & & & x_3 & & & & & - \; s_3 & + \; a & = 2, \\
& & & x_1, & & x_2, & & x_3, & & s_1, & & s_2, & s_3, & a & \geq 0.
\end{array}$$

(b) Express the phase-I objective as a function of the non-basic variables: $w = a = 2 - x_3 + s_3$. Simplex iterations (minimization):

$\mathbf{x}_B$	w	x_1	x_2	x_3	s_1	s_2	s_3	a	$\mathbf{B}^{-1}\mathbf{b}$	
w	1	0	0	1	0	0	−1	0	2	Entering variable: x_3 (red. cost: −1)
s_1	0	1	2	2	1	0	0	0	7	Ratio: $7/2 > 2/1$
s_2	0	2	1	1	0	1	0	0	6	Ratio: $6/1 > 2/1$
a	0	0	0	1	0	0	−1	1	2	Ratio: 2/1. Leaving variable: a
w	1	0	0	0	0	0	0	−1	0	All reduced costs ≥ 0
s_1	0	1	2	0	1	0	2	−2	3	$\Longrightarrow$ optimum of phase-I
s_2	0	2	1	0	0	1	1	−1	4	$\Longrightarrow$ feasible basis $\Longrightarrow$ phase-II
x_3	0	0	0	1	0	0	−1	1	2	

Express the phase-II objective as a function of the non-basic variables:
$z = x_2 + x_3 = x_2 + (2 + s_3)$. Simplex iterations (maximization):

$\mathbf{x}_B$	z	x_1	x_2	x_3	s_1	s_2	s_3	$\mathbf{B}^{-1}\mathbf{b}$	
z	1	0	−1	0	0	0	−1	2	Entering variable: x_2 (red. cost: 1)
s_1	0	1	2	0	1	0	2	3	Ratio: 3/2. Leaving variable: s_1
s_2	0	2	1	0	0	1	1	4	Ratio: $4/1 > 3/2$
x_3	0	0	0	1	0	0	−1	2	Ratio: denominator: 0
z	1	1/2	0	0	1/2	0	0	7/2	All reduced costs ≤ 0
x_2	0	1/2	1	0	1/2	0	1	3/2	$\Longrightarrow$ optimum phase-II
s_2	0	3/2	0	0	−1/2	1	0	5/2	
x_3	0	0	0	1	0	0	−1	2	

The problem is solved to optimality. Optimal solution: $x_1 = 0$, $x_2 = 3/2$, $x_3 = 2$, $z = 7/2$.

(c) Not all reduced costs of non-basic variables are negative. The reduced cost of s_3 equals 0. Continue the simplex iterations.

$\mathbf{x}_B$	z	x_1	x_2	x_3	s_1	s_2	s_3	$\mathbf{B}^{-1}\mathbf{b}$	
z	1	1/2	0	0	1/2	0	0	7/2	Entering variable: s_3 (red. cost: 0)
x_2	0	1/2	1	0	1/2	0	1	3/2	Ratio: (3/2)/1
s_2	0	3/2	0	0	−1/2	1	0	5/2	Ratio: denominator: 0
x_3	0	0	0	1	0	0	−1	2	Ratio: denominator: < 0
z	1	1/2	0	0	1/2	0	0	7/2	All reduced costs ≤ 0
s_3	0	1/2	1	0	1/2	0	1	3/2	$\Longrightarrow$ optimum phase-II
s_2	0	3/2	0	0	−1/2	1	0	5/2	
x_3	0	1/2	1	1	1/2	0	0	7/2	

The next iteration will lead to the previous basis, hence all optimal bases are found. Optimal bases: $(x_2, s_2, x_3) = (3/2, 5/2, 2)$ (with $x_1 = s_1 = s_3 = 0$) and $(s_3, s_2, x_3) = (3/2, 5/2, 7/2)$ (with $x_1 = x_2 = s_1 = 0$).

All optimal points in the (x_1, x_2, x_3) space are given by the convex hull of $(x_1, x_2, x_3) = (0, 3/2, 2)$ and $(x_1, x_2, x_3) = (0, 0, 7/2)$. The set of optimal solutions is thus expressed as $X^* = \{\alpha(0, 3/2, 2) + (1 - \alpha)(0, 0, 7/2) \,:\, \alpha \in [0, 1]\} = \{(0, 3\alpha/2, (7 - 3\alpha)/2) \,:\, \alpha \in [0, 1]\}$.

Solutions to Question 3

See the course book, Theorem 4.1 and its proof.

Solutions to Question 4

(a) Since all coefficients in the LHS of (1b) are even numbers, all coefficients in the inequality can be divided by 2 $\Longrightarrow 3x_1 + 1x_2 \leq 9.5 \Longrightarrow$ the cutting plane $3x_1 + 1x_2 \leq 9$

(b) The optimal tableau:

$\mathbf{x}_B$	z	x_1	x_2	s_1	s_2	$\mathbf{B}^{-1}\mathbf{b}$
z	1	0	0	$-9/14$	$-1/14$	$-7\frac{3}{14} = -101/14$
x_1	0	1	0	$5/14$	$-1/14$	$1\frac{11}{14} = 25/14$
x_2	0	0	1	$-1/14$	$3/14$	$3\frac{9}{14} = 51/14$

Gomory cut from the x_1-row: $x_1 + \frac{5}{14}s_1 - \frac{1}{14}s_2 = 1\frac{11}{14} \iff x_1 - s_2 - 1 = \frac{11}{14} - \frac{5}{14}s_1 - \frac{13}{14}s_2$.
As the LHS is integer for all feasible solutions and the RHS $\leq \frac{11}{14}$, the RHS must be ≤ 0.
Hence the inequality $x_1 - s_2 - 1 \leq 0$ must hold.
Substituting for $s_2 = 20 - x_1 - 5x_2$ yields the inequality $x_1 - (20 - x_1 - 5x_2) - 1 \leq 0 \iff$ $(1+1)x_1 + (5)x_2 \leq (0 + 20 + 1) \iff \boxed{2x_1 + 5x_2 \leq 21}$

Gomory cut from the x_2-row: $x_2 - \frac{1}{14}s_1 + \frac{3}{14}s_2 = 3\frac{9}{14} \iff x_2 - s_1 - 3 = \frac{9}{14} - \frac{13}{14}s_1 - \frac{3}{14}s_2$.
As the LHS is integer for all feasible solutions and the RHS $\leq \frac{9}{14}$, the RHS must be ≤ 0.
Hence the inequality $x_2 - s_1 - 3 \leq 0$ must hold.
Substituting for $s_1 = 9 - 3x_1 - x_2$ yields the inequality $x_2 - (9 - 3x_1 - x_2) - 3 \leq 0 \iff$ $(3)x_1 + (1+1)x_2 \leq (0 + 9 + 3) \iff \boxed{3x_1 + 2x_2 \leq 12}$

Solutions to Question 5

(a) Use Dijkstra's algorithm. ... calculations should be reported here ... Optimal path: $1 \longrightarrow 3 \longrightarrow 4$

(b) Optimal basis: $\mathbf{x}_B = (x_{12}, x_{13}, x_{34})$.
The three optimal paths are given by $1 \longrightarrow 2$, $1 \longrightarrow 3$, and $1 \longrightarrow 3 \longrightarrow 4$.

(c) For the optimal basis, given by $\mathbf{x}_B = (x_{12}, x_{13}, x_{34})$ and $\mathbf{x}_N = (x_{23}, x_{24})$, the following relations hold:

$$\mathbf{c}_N^\top = \begin{pmatrix} c_{23} & c_{24} \end{pmatrix} = \begin{pmatrix} 2+\delta_{23} & 7 \end{pmatrix},\ \mathbf{c}_B^\top = \begin{pmatrix} c_{12} & c_{13} & c_{34} \end{pmatrix} = \begin{pmatrix} 2 & 3+\delta_{13} & 5 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix},$$

$$\mathbf{B}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \text{ and } \mathbf{N} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Derivation of the reduced costs:

$$\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} = \begin{pmatrix} 2+\delta_{23} & 7 \end{pmatrix} - \begin{pmatrix} 2 & 3+\delta_{13} & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \end{pmatrix} = \ldots$$

$$= \begin{pmatrix} 1+\delta_{23}-\delta_{13} & 1-\delta_{13} \end{pmatrix} \geq \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

The current optimal solution, given by $x_{12} = 1$, $x_{13} = 2$, $x_{23} = 0$, $x_{24} = 0$, and $x_{34} = 1$, is optimal for any $(\delta_{13}, \delta_{23}) \in \mathbb{R}^2$ such that the inequalities $\delta_{13} \leq 1$ and $\delta_{13} - \delta_{23} \leq 1$ hold.