EXAM
MVE166/MVE165/MMG631
Linear and integer optimization with applications

- **Date:** 2025-01-07
  - **Hours:** 08:30–12:30
- **Aids:** Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3) points
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade
  - Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
- **Examiner:** Ann-Brith Strömberg
  - **Phone:** 0705-273645

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

[3p]

Modeling using integer linear optimization

The problem of finding a shortest (in terms of minimum cost) path from node s to node t in a directed graph $G = (N, A, \mathbf{d})$, where N denotes the set of nodes, $s, t \in N$, A denotes the set of directed arcs, $\mathbf{d} = [d_{ij}]_{(i,j)\in A}$, and $d_{ij} > 0$ denotes the length (cost) of the directed arc $(i, j) \in A$, can be modeled as to

$$
\begin{array}{llcl}
\text{minimize} & \displaystyle\sum_{(i,j)\in A} d_{ij}x_{ij}, & & \\
\text{subject to} & \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} & = & \begin{cases} -1, & k = s, \\ 1, & k = t, \\ 0, & k \in N \setminus \{s, t\}, \end{cases} \\
& x_{ij} & \geq & 0, \qquad (i, j) \in A,
\end{array}
$$

where the value of the variable x_{ij} represents the number of times that the directed arc $(i, j) \in A$ is to be traversed in the path found.

Consider an extension of the shortest path problem such that each node visit generates a profit. Specifically, assume that each node $i \in N \setminus \{s, t\}$ that is visited along a path generates a profit $q_i > 0$. Assume also that each node may be visited at most once.

Extend the model above such that also "node visiting profits" are accounted for in the objective function.

Define carefully all variables introduced. Moreover, all functions involved in the model must be linear.

Question 2

THE SIMPLEX ALGORITHM, FEASIBILITY, AND OPTIMALITY

Consider the linear optimization problem to

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + 3x_2 - 3x_3, && \text{(1a)}\\
\text{subject to} \quad & x_1 + 2x_2 + x_3 \leq 8 && \text{(1b)}\\
& 2x_1 + x_2 - x_3 \geq 3 && \text{(1c)}\\
& x_1, \quad x_2, \quad x_3 \geq 0. && \text{(1d)}
\end{aligned}
$$

(a) **[1p]**

Reformulate the problem (1) such that it can be solved using the 2-phase simplex method.

(b) **[1p]**

Solve the reformulated problem from (a) using the simplex method, phase 1 and phase 2.

(c) **[1p]**

Verify that the solution found in (b) is optimal to (1), by using linear optimization duality and complementarity relations.

Question 3

SHADOW PRICES

Consider the linear optimization problem

$$
\begin{aligned}
z^* := \max \quad & 2x_1 + 3x_2, && \text{(2a)}\\
\text{s.t.} \quad & x_1 + x_2 \leq 5, && \text{(2b)}\\
& 2x_1 + 5x_2 \leq 20, && \text{(2c)}\\
& x_1, \quad x_2 \geq 0. && \text{(2d)}
\end{aligned}
$$

(a) **[1p]**

Compute the shadow price for each of the two inequality constraints (2b) and (2c).

(b) **[2p]**

Utilize the shadow prices computed in (a) to predict the optimal value of (2) for the case when the right–hand–sides are altered from $\begin{pmatrix} 5 \\ 20 \end{pmatrix}$ to $\begin{pmatrix} 5 \\ 20 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 21 \end{pmatrix}$. Then, show whether or not the computed shadow prices are valid for this particular change of the values of the right–hand–sides.

Question 4

[3p]

Optimal basic solutions

For a general primal–dual pair of linear optimization problems given by

$$\begin{array}{llll} (\mathrm{P}) & z^* := & \max\{\, \mathbf{c}_B^\top \mathbf{x}_B + \mathbf{c}_N^\top \mathbf{x}_N \;:\; \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b};\; \mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0} \,\} \\ (\mathrm{D}) & w^* := & \min\{\, \mathbf{b}^\top \mathbf{y} \;:\; \mathbf{B}^\top \mathbf{y} \geq \mathbf{c}_B;\; \mathbf{N}^\top \mathbf{y} \geq \mathbf{c}_N;\; \mathbf{y} \text{ free} \,\}, \end{array}$$

Theorem 6.4 states that if $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, $\mathbf{x}_N = \mathbf{0}$ is an optimal basic solution to (P) then $\mathbf{y}^\top = \mathbf{c}_B^\top \mathbf{B}^{-1}$ is optimal in (D) and it holds that $z^* = w^*$.

Prove this theorem.

Question 5

Finding multiple optimal solutions

Consider the following binary linear optimization problem:

$$\begin{array}{lrcrcrcrcll} \text{maximize} & 5x_1 & + & 3x_2 & + & 3x_3 & + & x_4, & & & \text{(3a)} \\ \text{subject to} & 7x_1 & + & 5x_2 & + & 6x_3 & + & 3x_4 & \leq & 14 & \text{(3b)} \\ & x_1, & & x_2, & & x_3, & & x_4 & \in & \{0,1\}. & \text{(3c)} \end{array}$$

(a) **[1.5p]**

Solve the problem (3) using the branch–and–bound algorithm, where the relaxation is defined by relaxing the integrality constraints on the variables. Use breadth–first search and search the 0-branch first. Terminate the algorithm when an optimal solution is verified. State the optimal solution and motivate why it is optimal in (3).

(b) **[1.5p]**

Here, you should describe how to investigate whether an optimal solution found is a *unique* optimal solution to the binary linear optimization problem (3). This can be done by adding one or several linear constraints to the model.

Given an optimal solution to the problem (3)—e.g., the solution found in (a)—formulate a linear constraint such that the properties of an optimal solution to the resulting problem, in which this new constraint is added to (3), yields information about whether or not the solution from (a) is a unique optimal solution to (3).

Explain your reasoning carefully.

You do *not* have to solve the resulting problem.

Solution proposals to EXAM 2025-01-07

MVE166/MVE165/MMG631 Linear and integer optimization with applications

These solutions may be brief in relation to the requirements on your answers, in particular regarding motivations.

Solutions to Question 1

Let $u_k = 1$ if node k is visited, otherwise $u_k = 0$, $k \in N \setminus \{s,t\}$.

$$
\begin{array}{lrcl}
\text{minimize} & \displaystyle\sum_{(i,j)\in A} d_{ij}x_{ij} - \sum_{i\in N\setminus\{s,t\}} q_i u_i, & & \\
\text{subject to} & \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} & = & \begin{cases} -1, & k = s, \\ 1, & k = t, \\ 0, & k \in N \setminus \{s,t\}, \end{cases} \\
& \displaystyle\sum_{i:(i,k)\in A} x_{ik} & = & u_k, \quad k \in N \setminus \{s,t\} \\
& x_{ij} & \geq & 0, \quad (i,j) \in A, \\
& u_k & \in & \{0,1\}, \quad k \in N \setminus \{s,t\}
\end{array}
$$

Solutions to Question 2

(a) Add slack- and surplus variables, s_1 and s_2, and an artificial variable, a, and replace the objective by the phase 1 objective to minimize a:

$$
\begin{array}{lrcrcrcrcrcrcl}
\text{minimize} & & & & & & & & & & & a, & & \\
\text{subject to} & x_1 & + & 2x_2 & + & x_3 & + & s_1 & & & & & = & 8 \\
& 2x_1 & + & x_2 & - & x_3 & & & - & s_2 & + & a & = & 3 \\
& x_1, & & x_2, & & x_3, & & s_1, & & s_2, & & a & \geq & 0.
\end{array}
$$

Solve this problem. At an optimal basis, if $a = 0$ then remove the variable a, reinstate the original objective, and solve from the basis that was found as optimal in the phase 1 problem. If $a > 0$ at optimum of phase 1, then the problem (1) has no feasible solution.

(b) Solution of the phase 1 problem, with the objective to minimize $w = a$: An initial basis is $B = (s_1, a)$ with $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{B}^{-1}$, $\mathbf{c}_B = (0,1)$, $N = (x_1, x_2, x_3, s_2)$, $\mathbf{N} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & -1 \end{pmatrix}$, $\mathbf{c}_N = (0,0,0,0)$, and the reduced costs $\bar{\mathbf{c}}_N^\top = \mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} = (-2,-1,1,1)$.

$\mathbf{x}_B$	w	x_1	x_2	x_3	s_1	s_2	a	$\mathbf{B}^{-1}\mathbf{b}$
w	1	**2**	1	-1	0	-1	0	3
s_1	0	1	2	1	1	0	0	8
a	0	**2**	1	-1	0	-1	1	3
w	1	0	0	0	0	0	-1	0
s_1	0	0	3/2	3/2	1	1/2	-1/2	13/2
x_1	0	1	1/2	-1/2	0	-1/2	1/2	3/2

The solution $\mathbf{x} = (3/2, 0, 0)^\top$ is feasible in (1). The corresponding values of the slack, surplus, and artificial variables are $s_1 = 13/2$, $s_2 = 0$, and $a = 0$, respectively.

Solution of the phase 2 problem, with the objective to minimize $z = 2x_1 + 3x_2 - 3x_3$: An initial basis is is given by $B = (s_1, x_1)$ with $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $\mathbf{B}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{c}_B = (0, 2)$, $N = (x_2, x_3, s_2)$, $\mathbf{N} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$, $\mathbf{c}_N = (3, -3, 0)$, and the reduced costs $\bar{\mathbf{c}}_N^\top = \mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} = (2, -2, 1)$.

$\mathbf{x}_B$	z	x_1	x_2	x_3	s_1	s_2	$\mathbf{B}^{-1}\mathbf{b}$
z	1	0	**-2**	2	0	-1	3
s_1	0	0	3/2	3/2	1	1/2	13/2
x_1	0	1	**1/2**	-1/2	0	-1/2	3/2
z	1	4	0	0	0	**-3**	9
s_1	0	-3	0	3	1	**2**	2
x_2	0	2	1	-1	0	-1	3
z	1	**-1/2**	0	9/2	3/2	0	12
s_2	0	-3/2	0	3/2	1/2	1	1
x_2	0	**1/2**	1	1/2	1/2	0	4
z	1	0	1	5	2	0	16
s_2	0	0	3	3	2	1	13
x_1	0	1	2	1	1	0	8

The solution $\mathbf{x} = (8, 0, 0)^\top$ is optimal in (1). The optimal values of the slack and surplus variables are $s_1 = 0$ and $s_2 = 13$, respectively.

(c) The LP dual of (1) is given by

$$
\begin{array}{lrcrcr}
\text{minimize} & 8y_1 & + & 3y_2, & & \\
\text{subject to} & y_1 & + & 2y_2 & \geq & 2, \\
& 2y_1 & + & y_2 & \geq & 3, \\
& y_1 & - & y_2 & \geq & -3, \\
& y_1 & & & \geq & 0, \\
& & & y_2 & \leq & 0.
\end{array}
$$

Complementarity: For any $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ that are feasible in the primal and dual problems, respectively, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are optimal in their respective problems if and only if the two equalities $\sum_{i=1}^{2} \hat{y}_i(b_i - \mathbf{A}_i\hat{\mathbf{x}}) = 0$ and $\sum_{j=1}^{3} \hat{x}_j(c_j - \mathbf{A}_j^\top\hat{\mathbf{y}}) = 0$ hold.

For $\hat{\mathbf{x}} = (8, 0, 0)^\top$ it holds that $\mathbf{b} - \mathbf{A}\hat{\mathbf{x}} = \begin{pmatrix} 8 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 8-8 \\ -3+2\cdot 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \end{pmatrix}$, such that $\hat{y}_1 \cdot 0 + \hat{y}_2 \cdot 13 = 0 \Longrightarrow \hat{y}_2 = 0$.

Since $\hat{x}_1 = 8 > 0$ and $\hat{x}_2 = \hat{x}_3 = 0$, it must hold that $\hat{x}_1(c_1 - \mathbf{A}_1^\top\hat{\mathbf{y}}) = 8(2 - \hat{y}_1 + 2\hat{y}_2) = 0 \Longrightarrow \hat{y}_1 - 2\hat{y}_2 = 2$. Since $\hat{y}_2 = 0$ it follows that $\hat{y}_1 = 2$.

By strong duality, the equality $\mathbf{c}^\top\hat{\mathbf{x}} = \mathbf{b}^\top\hat{\mathbf{y}}$ should hold.
Check: $\mathbf{c}^\top\hat{\mathbf{x}} = 2\cdot 8 + 3\cdot 0 - 3\cdot 0 = 16$ and $\mathbf{b}^\top\hat{\mathbf{y}} = 8\cdot 2 - 3\cdot 0 = 16$.

Solutions to Question 3

(a) Solving the primal problem graphically gives the optimal solution $\mathbf{x}^* = \frac{1}{3}\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ with the optimal basis $(x_1, x_2)^\top$, $\mathbf{c}_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}$, and $\mathbf{B}^{-1} = \frac{1}{3}\begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix}$.

Shadow price = Optimal dual solution: $\mathbf{y}^\top = \mathbf{c}_B^\top \mathbf{B}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 & 1 \end{pmatrix}$

(b) Alter the RHSs of the inequality constraints to $\mathbf{b}^{\text{new}} = \begin{pmatrix} 4 \\ 21 \end{pmatrix}$. A reasonable prediction of the optimal value, given this new RHS is given by $\mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}^{\text{new}} = \frac{1}{3}\begin{pmatrix} 4 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 21 \end{pmatrix} = \frac{1}{3}(4 \cdot 4 + 1 \cdot 21) = \frac{37}{3}$.

The shadow prices are valid for all RHS such that $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$. For $\mathbf{b}^{\text{new}} = \begin{pmatrix} 4 \\ 21 \end{pmatrix}$ it holds that $\mathbf{B}^{-1}\mathbf{b}^{\text{new}} = \frac{1}{3}\begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 21 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 \cdot 4 - 1 \cdot 21 \\ -2 \cdot 4 + 1 \cdot 21 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 13 \end{pmatrix} \not\geq \mathbf{0}$. Since the current basis in not feasible for the altered RHS, the prediction of the optimal value is not valid. The prediction is optimistic, such that the optimal value of the altered problem is $< \frac{37}{3}$.

Solutions to Question 4

(Lundgren et al., proof of Theorem 6.4)

For a pair of primal and dual solutions to be optimal in the respective linear optimization problems, it must hold that (i) the solutions are feasible in their respective problems and that (ii) their objective values are equal.

(i): Since $(\mathbf{x}_B, \mathbf{x}_N) = (\mathbf{B}^{-1}\mathbf{b}, \mathbf{0})$ is an optimal basic solution to (P) it is also feasible in (P). For $\mathbf{y}^\top = \mathbf{c}_B^\top \mathbf{B}^{-1}$ it holds that $(\mathbf{B}^\top \mathbf{y})^\top - \mathbf{c}_B^\top = \mathbf{y}^\top \mathbf{B} - \mathbf{c}_B^\top = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{B} - \mathbf{c}_B^\top = \mathbf{c}_B^\top - \mathbf{c}_B^\top = \mathbf{0}^\top \geq \mathbf{0}^\top$, such that the first set of constraints in (D) are fulfilled. Further, it holds that $(\mathbf{N}^\top \mathbf{y})^\top - \mathbf{c}_N^\top = \mathbf{y}^\top \mathbf{N} - \mathbf{c}_N^\top = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^\top$. Since $(\mathbf{x}_B, \mathbf{x}_N) = (\mathbf{B}^{-1}\mathbf{b}, \mathbf{0})$ is assumed to be an optimal basic solution to (P) it must hold that the reduced costs of the nonbasic variables are nonpositive, i.e., that $\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} \leq \mathbf{0}^\top$. It hence follows that also the second set of constraints in (D) are fulfilled.

(ii): Since $z^* = \mathbf{c}_B^\top \mathbf{x}_B + \mathbf{c}_N^\top \mathbf{x}_N = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b} + 0 = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}$ and $w^* = \mathbf{b}^\top \mathbf{y} = \mathbf{b}^\top (\mathbf{c}_B^\top \mathbf{B}^{-1})^\top = \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{b}$ hold, it follows that $z^* = w^*$, which concludes the proof.

Solutions to Question 5

(a) Relaxing the binary constraints yields the continuous knapsack problem to

$$\begin{array}{lccccccccc} \text{maximize} & 5x_1 & + & 3x_2 & + & 3x_3 & + & x_4, & & \\ \text{subject to} & 7x_1 & + & 5x_2 & + & 6x_3 & + & 3x_4 & \leq & 14. \\ & x_1, & & x_2, & & x_3, & & x_4 & \in & [0,1]. \end{array}$$

This problem can be solved as follows:

1. Sort the ratios $\frac{c_j}{a_j}$ in descending order, where c_j and a_j denote the objective and constraint coefficient, respectively, of the variable x_j: $\left\{\frac{5}{7}, \frac{3}{5}, \frac{3}{6}, \frac{1}{3}\right\} \approx \{0.714, 0.6, 0.5, 0.333\}$ yields the order x_1, x_2, x_3, x_4.
2. For some $k \in \{2,3\}$, set $x_j = 1$ for $j = 1, \dots, k-1$, $x_k = \frac{1}{a_k}\left(14 - \sum_{j=1}^{k-1} a_j\right) \in [0,1)$, $x_j = 0$, $j = k+1, \dots, 4$.

Branch–and–bound: Branch on the fractional variable.

P0: $\mathbf{x} = (1, 1, \frac{1}{6}(14-12), 0) = (1, 1, \frac{1}{3}, 0)$, $\overline{z} = 5 + 3 + \frac{3}{3} = 9$. Branch on $x_3 \implies$ P1&P2
P1($x_3 = 0$): $\mathbf{x} = (1, 1, 0, \frac{1}{3}(14-12)) = (1, 1, 0, \frac{2}{3})$, $\overline{z} = 5 + 3 + \frac{2}{3} = 8.67$. Branch on $x_4 \implies$ P3&P4
P2($x_3 = 1$): $\mathbf{x} = (1, \frac{1}{5}(14-13), 1, 0) = (1, \frac{1}{5}, 1, 0)$, $\overline{z} = 5 + \frac{3}{5} + 3 = 8.6$. Branch on $x_2 \implies$ P5&P6
P3($x_3 = 0$, $x_4 = 0$): $\mathbf{x} = (1, 1, 0, 0)$, $\underline{z} = 5 + 3 = 8$. Prune the branch
P4($x_3 = 0$, $x_4 = 1$): prune the branch
P5($x_3 = 1$, $x_2 = 0$): prune the branch
P6($x_3 = 1$, $x_2 = 1$): prune the branch

Since the upper bounds in all (two) branches (i.e., P1 and P2) are < 9, and since we have found a feasible solution $\mathbf{x} = (1, 1, 0, 0)$ in P3, with objective value 8, we conclude that the latter is an optimal solution. We can thus prune all branches.

(b) An optimal solution is given by $\bar{\mathbf{x}} = (1, 1, 0, 0)^\top$. A constraint that excludes this solution, but no other feasible solutions, is given by $x_1 + x_2 \leq 1$. Hence, state the following problem

$$\begin{array}{lccccccccc} \text{maximize} & 5x_1 & + & 3x_2 & + & 3x_3 & + & x_4, & & & (4a) \\ \text{subject to} & 7x_1 & + & 5x_2 & + & 6x_3 & + & 3x_4 & \leq & 14, & (4b) \\ & x_1 & + & x_2 & & & & & \leq & 1, & (4c) \\ & x_1, & & x_2, & & x_3, & & x_4 & \in & \{0,1\}. & (4d) \end{array}$$

If an optimal solution to the problem (4) has an objective value ≤ 7, then the solution $\mathbf{x} = (1, 1, 0, 0)$ is the unique optimal solution to the problem (3). If there is an optimal solution to (4) with objective value $= 8$, it must be an alternative optimal solution to (3), since the solution $\mathbf{x} = (1, 1, 0, 0)$ is not feasible in (4).