EXAM
MVE166/MVE165/MMG631
Linear and integer optimization with applications

- **Date:** 2024-08-29
  - **Hours:** 14:00–18:00
- **Aids:** Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3) points
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade
  - Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
- **Examiner:** Ann-Brith Strömberg **Phone:** 0705-273645

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

[3p]

In bio-fuel depots the materials are stored in several piles within designated areas. A depot is usually several hundred meters in both length and width. Since there is always a risk of thermal fire in such a depot, a fire detection system is needed. Such a system can be set up using thermal cameras placed at several locations. We wish to place thermal cameras at certain locations such that the whole depot is continuously monitored via the cameras. There is, however, an upper limit on the number of cameras placed in the bio-fuel depot, and we wish to minimize the cost for setting up the system of cameras.

Assume that the depot is partitioned into m sub-areas, denoted by S_i, $i = 1, \dots, m$. There are n potential locations for the cameras, denoted by L_j, $j = 1, \dots, n$. For each location L_j and each sub-area S_i, we define the parameter

$$a_{ij} = \begin{cases} 1, & \text{if a camera at location } L_j \text{ can monitor sub-area } S_i, \\ 0, & \text{otherwise,} \end{cases} \qquad i = 1, \dots, m, \; j = 1, \dots, n.$$

Denote by k the upper limit on the number of cameras placed in the depot, where $k < n$. Further, let $c_j > 0$ represent the cost of setting up a thermal camera at location L_j, $j = 1, \dots, n$.

Formulate an integer linear optimization model of the problem to choose locations for cameras, such that all sub-areas are monitored and such that the total cost of the system of cameras is as low as possible.

Question 2

To determine whether or not there exists a feasible solution to the system

$$
\begin{aligned}
2x_1 &+ 3x_2 &- x_3 &+ 2x_4 &= 3, && \text{(1a)}\\
x_1 &+ x_2 &- 2x_3 &+ 2x_4 &= 5, && \text{(1b)}\\
x_1, & \quad x_2, & x_3, & \quad x_4 &\geq 0. && \text{(1c)}
\end{aligned}
$$

one can introduce the artificial variables a_1 and a_2, and solve the phase-I problem

$$
\begin{aligned}
\min \quad w = & & & & & a_1 &+ a_2, & & && \text{(2a)}\\
\text{s.t} \quad & 2x_1 &+ 3x_2 &- x_3 &+ 2x_4 &+ a_1 & &= 3, && \text{(2b)}\\
& x_1 &+ x_2 &- 2x_3 &+ 2x_4 & &+ a_2 &= 5, && \text{(2c)}\\
& x_1, & x_2, & x_3, & x_4, & a_1, & a_2 &\geq 0. && \text{(2d)}
\end{aligned}
$$

(a) **[1p]**

State the linear optimization dual of the problem (2).

(b) **[1p]**

Solve the dual problem graphically.

(c) **[1p]**

Use the dual optimal solution to determine whether there exists a feasible solution to the system (1).

Question 3

Consider the integer linear optimization problem

$$
\begin{aligned}
z^* := \max \quad z = & \quad x_1 &+ 2x_2 & & && \text{(3a)}\\
\text{s.t} \quad & -3x_1 &+ 4x_2 &\leq 4, && \text{(3b)}\\
& 4x_1 &+ 6x_2 &\leq 23, && \text{(3c)}\\
& 2x_1 &- x_2 &\leq 5, && \text{(3d)}\\
& x_1, & x_2 &\geq 0 \text{ and integer.} && \text{(3e)}
\end{aligned}
$$

(a) **[2p]**

Solve the problem (3) using the Branch–and–Bound method. Illustrate your computations in a search tree. The node subproblems may be solved graphically.

State and verify the optimal solution found using theory from the course.

(b) **[1p]**

Give the convex hull of the feasible set defined by the constraints (3b)–(3e), either in terms of linear constraints or as convex combinations of certain points.

Question 4

Consider the linear optimization problem to

$$\begin{aligned}
\text{maximize} \quad z = \; & 4x_1 & & & + \; & x_3, & & & \quad (4a)\\
\text{s.t} \quad & x_1 & + \; & 3x_2 & + \; & x_3 & \leq \; & 4, & \quad (4b)\\
& x_1 & - \; & x_2 & & & \leq \; & 2, & \quad (4c)\\
& x_1, & & x_2, & & x_3 & \geq \; & 0. & \quad (4d)
\end{aligned}$$

(a) **[2p]**

Solve this problem using the simplex method and compute **all** optimal solutions to the problem.

(b) **[1p]**

Assume that the right–hand–side vector of the inequality constraints (4b)–(4c) is changed from $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 4+\delta_1 \\ 2+\delta_2 \end{pmatrix}$. For what values of the vector $\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \in \mathbb{R}^2$ will **all** of the optimal bases in (a) stay feasible?

Question 5

[3p]

Consider the following linear optimization problem:

$$\begin{aligned}
z^* := \max_{\mathbf{x}\in\mathbb{R}^n} \quad & \mathbf{c}^\top \mathbf{x},\\
\text{s.t.} \quad & A\mathbf{x} \leq \mathbf{b},\\
& \mathbf{x} \geq \mathbf{0},
\end{aligned}$$

where $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$. Suppose that the feasible set $\{\, \mathbf{x} \geq \mathbf{0} \,|\, A\mathbf{x} \leq \mathbf{b} \,\}$ is nonempty and that also the feasible set of its linear optimization dual is nonempty.

Formulate this corresponding linear optimization dual problem and show that *weak duality* (Theorem 6.1 in the course book) holds between the primal and dual problems.

Solution proposals to EXAM 2024-08-29

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These solutions may be brief in relation to the requirements on your answers, in particular regarding motivations.

Solutions to Question 1

Define the decision variables as

$$x_j = \begin{cases} 1, & \text{if a camera is placed at location } L_j, \\ 0, & \text{otherwise,} \end{cases} \qquad j = 1, \dots, n.$$

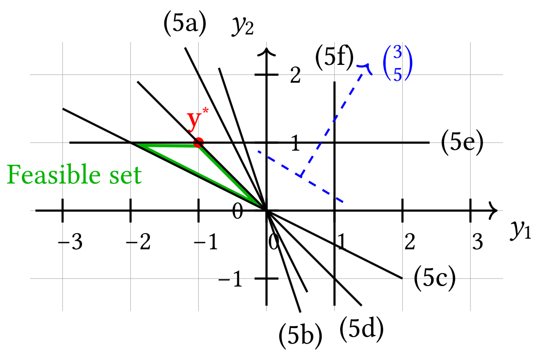
The problem is modelled as a set covering problem, i.e.,

$$\begin{array}{llcll} \min & \sum_{j=1}^{n} c_j x_j, & & & \\ \text{s.t} & \sum_{j=1}^{n} a_{ij} x_j & \geq & 1, & i = 1, \dots, m, \\ & \sum_{j=1}^{n} x_j & \leq & k, & \\ & x_j & \in & \{0, 1\}, & j = 1, \dots, n. \end{array}$$

Solutions to Question 2

(a)

$$\begin{array}{llrcrcll} \text{LP dual:} & w^* = \max & 3y_1 & + & 5y_2, & & & \\ & \text{s.t} & 2y_1 & + & y_2 & \leq & 0, & \text{(5a)} \\ & & 3y_1 & + & y_2 & \leq & 0, & \text{(5b)} \\ & & -y_1 & - & 2y_2 & \leq & 0, & \text{(5c)} \\ & & 2y_1 & + & 2y_2 & \leq & 0, & \text{(5d)} \\ & & y_1 & & & \leq & 1, & \text{(5e)} \\ & & & & y_2 & \leq & 1, & \text{(5f)} \end{array}$$



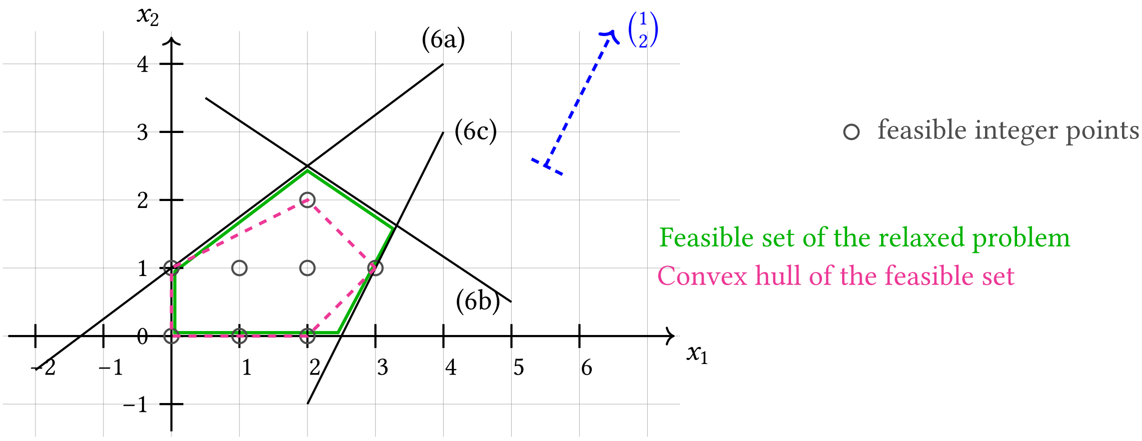
(b)

(c) $\mathbf{y}^* = (-1, 1)^\top$, $w^* = 2 > 0 \Longrightarrow$ In an optimal solution to the problem (2) at least one of the variables a_1 and a_2 must take a strictly positive value. Hence the system (1) has no feasible solution. By complementarity, since the constraint $y_2 \le 1$ ($y_1 \le 1$) is binding (not binding) at $\mathbf{y}^*$, the artificial variable a_2 (a_1) can (cannot) take a non-zero value at optimum of the problem (2).

Solutions to Question 3

(a) Relaxation of the integrality constraints results in the following LP:

$$\begin{array}{llrcrcrl} \max & z = & x_1 & + & 2x_2, & & & \\ \text{s.t} & & -3x_1 & + & 4x_2 & \le & 4, & \text{(6a)} \\ & & 4x_1 & + & 6x_2 & \le & 23, & \text{(6b)} \\ & & 2x_1 & - & x_2 & \le & 5, & \text{(6c)} \\ & & x_1, & & x_2 & \ge & 0. & \end{array}$$



(An illustration of the search tree should be included.)

(P0): $\bar{\mathbf{x}} = (2, 2.5)^\top$, $\bar{z} = 7 \Longrightarrow \boxed{z^* \le 7}$
Branching: $x_2 \le 2 \Longrightarrow$ (P1); $x_2 \ge 3 \Longrightarrow$ (P2)

(P1): $\bar{\mathbf{x}} = (2.75, 2)^\top$, $\bar{z} = 6.75 \Longrightarrow$ the upper bound on z in this branch is $\lfloor 6.75 \rfloor = 6$
Branching: $x_1 \le 2 \Longrightarrow$ (P3); $x_1 \ge 3 \Longrightarrow$ (P4)

(P2): infeasible, the node is pruned

(P3): $\underline{\mathbf{x}} = (2, 2)^\top$, $\underline{z} = 6 \Longrightarrow \boxed{z^* \ge 6}$
integer solution $\Longleftrightarrow$ candidate for optimal solution, the node is pruned

(P4): The node can be pruned, since the upper bound on z in this branch (from (P1)) is $\lfloor 6.75 \rfloor = 6$, and $z^* \ge 6$ (from (P3)).

An optimal solution is $\mathbf{x}^* = (2, 2)^\top$. The optimal value is $z^* = 6$.

(b) The convex hull is given by $\text{conv}\left\{ \binom{0}{0}, \binom{0}{1}, \binom{2}{2}, \binom{3}{1}, \binom{2}{0} \right\}$
$= \left\{ \mathbf{x} = \alpha_1 \binom{0}{0} + \alpha_2 \binom{0}{1} + \alpha_3 \binom{2}{2} + \alpha_4 \binom{3}{1} + \alpha_5 \binom{2}{0} \mid \sum_{k=1}^{5} \alpha_k = 1,\ \alpha_k \ge 0, k = 1, \dots, 5 \right\}$

Solutions to Question 4

(a)

$\mathbf{x}_B$	z	x_1	x_2	x_3	s_1	s_2	$\mathbf{B}^{-1}\mathbf{b}$	
z	1	**-4**	0	-1	0	0	0	
s_1	0	1	3	1	1	0	4	
s_2	0	**1**	-1	0	0	1	2	
z	1	0	**-4**	-1	0	4	8	
s_1	0	0	**4**	1	1	-1	2	
x_1	0	1	-1	0	0	1	2	
z	1	0	0	**0**	1	3	10	optimal basis B_1
x_2	0	0	1	**1/4**	1/4	-1/4	1/2	
x_1	0	1	0	1/4	1/4	3/4	5/2	
z	1	0	0	0	1	3	10	optimal basis B_2
x_3	0	0	4	1	1	-1	2	
x_1	0	1	-1	0	0	1	2	

There are two optimal bases: $\mathbf{x}_{B_1} = \binom{x_2}{x_1}$ and $\mathbf{x}_{B_2} = \binom{x_3}{x_1}$ corresponding to the two extreme points (optimal basic solutions) $\mathbf{x}^{1*} = \begin{pmatrix} 5/2 \\ 1/2 \\ 0 \end{pmatrix}$ and $\mathbf{x}^{2*} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

The set of all optimal solutions is defined as all convex combinations of these two points:

$$\mathbf{x}^* \in \left\{ \begin{pmatrix} 5\alpha/2 + 2(1-\alpha) \\ \alpha/2 \\ 2(1-\alpha) \end{pmatrix} \;:\; 0 \le \alpha \le 1 \right\} = \left\{ \begin{pmatrix} 2 + \alpha/2 \\ \alpha/2 \\ 2 - 2\alpha \end{pmatrix} \;:\; 0 \le \alpha \le 1 \right\}$$

(b) For the basis B_1 to stay feasible, it must hold that $B_1^{-1}\begin{pmatrix} 4+\delta_1 \\ 2+\delta_2 \end{pmatrix} \ge \mathbf{0}^2$.

$$B_1^{-1}\begin{pmatrix} 4+\delta_1 \\ 2+\delta_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 5/2 \end{pmatrix} + \begin{pmatrix} 1/4 & -1/4 \\ 1/4 & 3/4 \end{pmatrix}\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 1/2 + 1/4\delta_1 - 1/4\delta_2 \\ 5/2 + 1/4\delta_1 + 3/4\delta_2 \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\iff \delta_1 - \delta_2 \ge -2$ and $\delta_1 + 3\delta_2 \ge -10$

For the basis B_2 to stay feasible, it must hold that $B_2^{-1}\begin{pmatrix} 4+\delta_1 \\ 2+\delta_2 \end{pmatrix} \ge \mathbf{0}^2$.

$$B_2^{-1}\begin{pmatrix} 4+\delta_1 \\ 2+\delta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 2 + \delta_1 - \delta_2 \\ 2 + \delta_2 \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\iff \delta_1 - \delta_2 \ge -2$ and $\delta_2 \ge -2$.

Hence, for both bases to stay feasible, the following inequalities must hold (note that the third constraint is redundant):

$$\begin{aligned} \delta_1 &- \delta_2 &\ge -2, \\ & \delta_2 &\ge -2, \\ \delta_1 &+ 3\delta_2 &\ge -10. \end{aligned}$$

Solutions to Question 5

See the course book, proof of Theorem 6.1.